The Eigenvector-Eigenvalue Identity + other pragmatic topics in linear algebra for physicists

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NextGenerationEU

arXiv:1907.02534
1908.03795
2212.12565

Wikipedia:

Eigenvector-eigenvalue identity [edit]

For a Hermitian matrix A, the norm squared of the α-th component of a normalized eigenvector can be calculated using only the matrix eigenvalues and the eigenvalues of the corresponding minor matrix,

$$|v_{i\alpha}|^2 = \frac{\prod_k (\lambda_i(A) - \lambda_k(A_\alpha))}{\prod_{k\neq i} (\lambda_i(A) - \lambda_k(A))},$$

where A_α is the submatrix formed by removing the α-th row and column from the original matrix.[33][34][35] This identity also extends to diagonalizable matrices, and has been rediscovered many times in the literature.[34][36]

How to Solve ?

nxn Hamiltonian

Eigenvalue

$$H v_i = \lambda_i \, v_i$$

(no sum of i)

nx1 Eigenvector: $v_{\alpha i}$

Why ?

Neutrino Propagation in Matter
Including oscillations

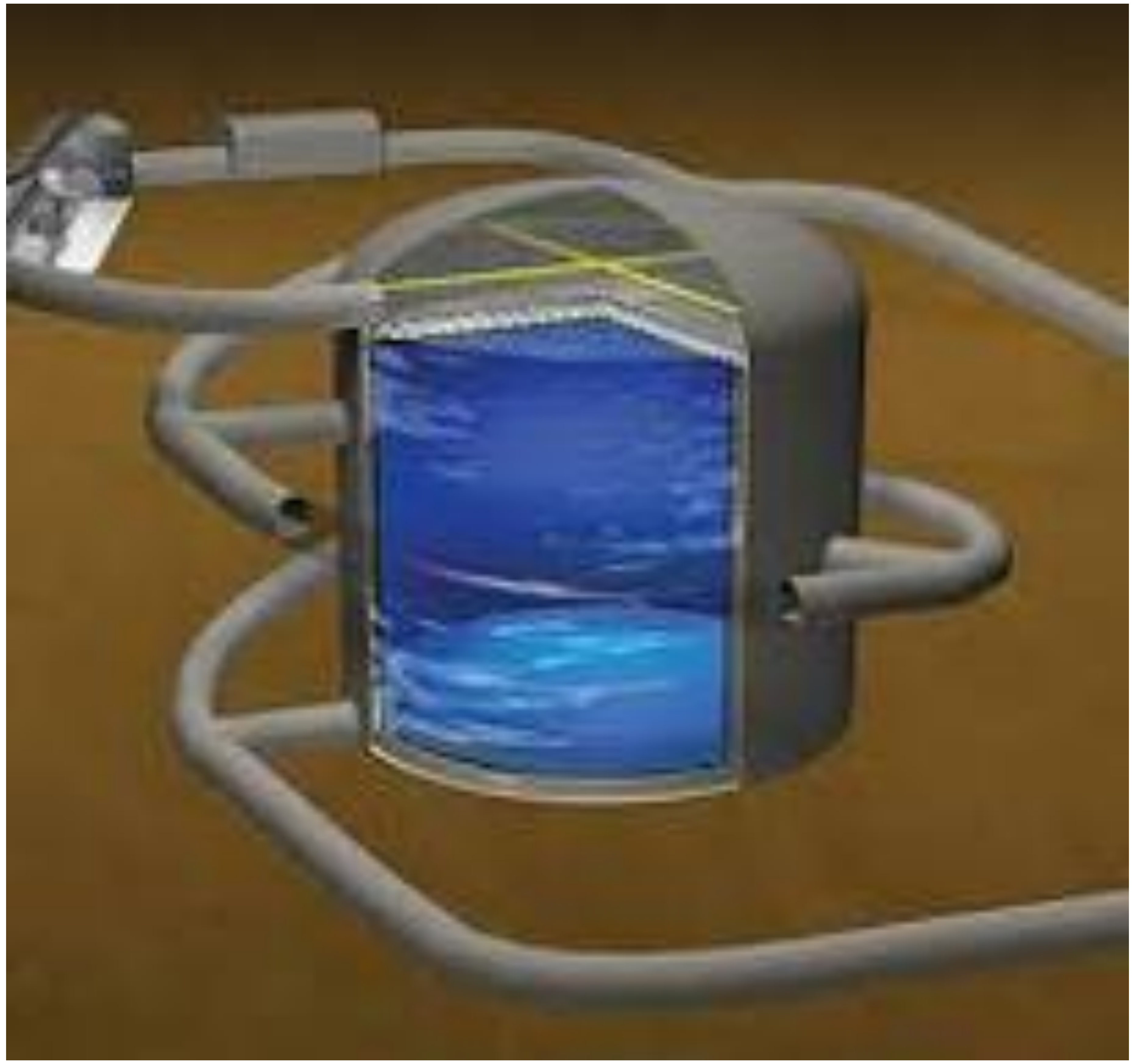
HyperK (260 kt Water Cherenkov)

$\nu_\mu \rightarrow \nu_e, \quad \bar{\nu}_\mu \rightarrow \bar{\nu}_e$

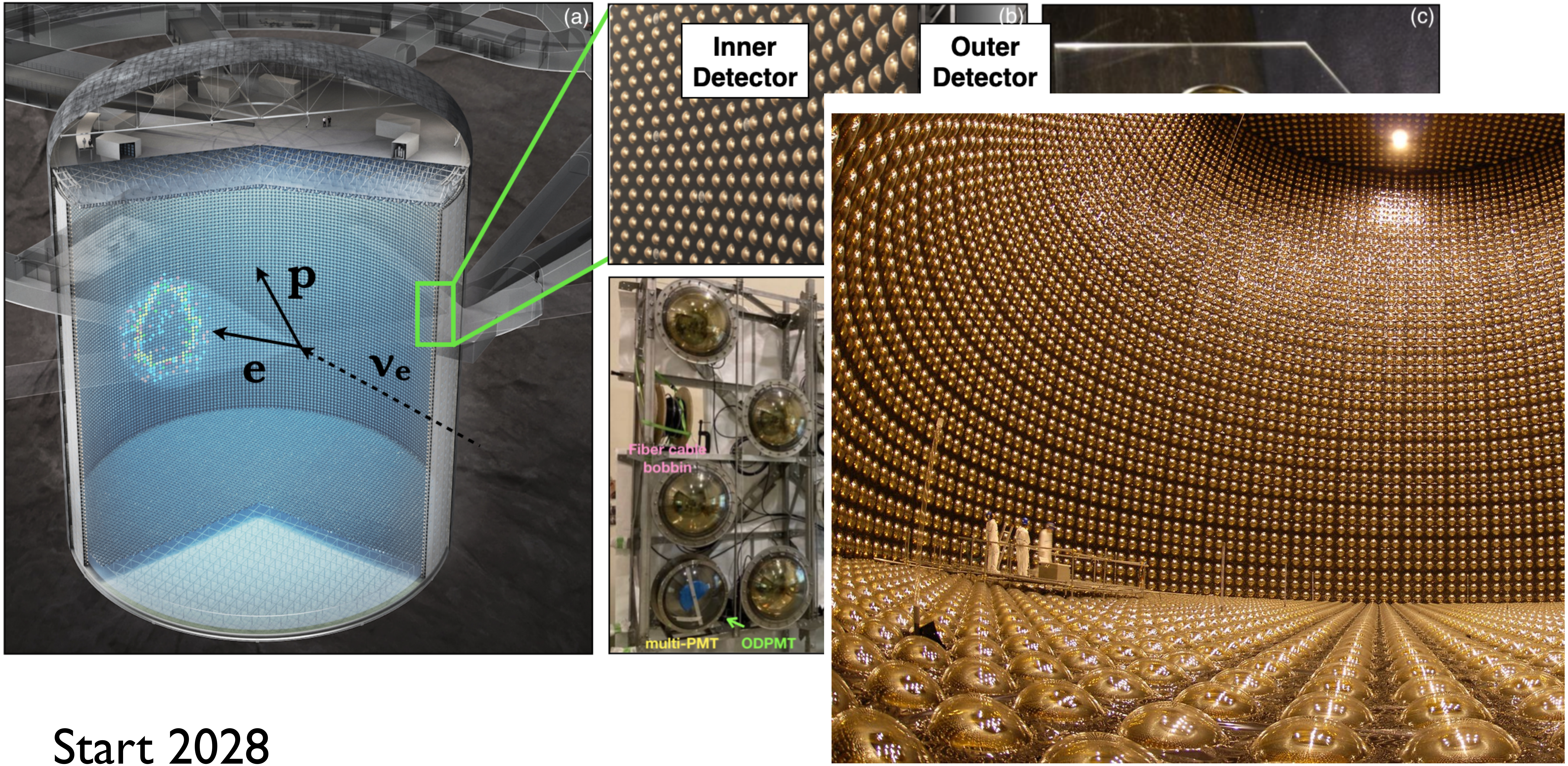
$\nu_\mu \rightarrow \nu_\mu, \quad \bar{\nu}_\mu \rightarrow \bar{\nu}_\mu$

295 km

1.3 MW



Water Cherenkov Detector like SuperK (50 kt) and KamiokaNDE (4.5 kt)

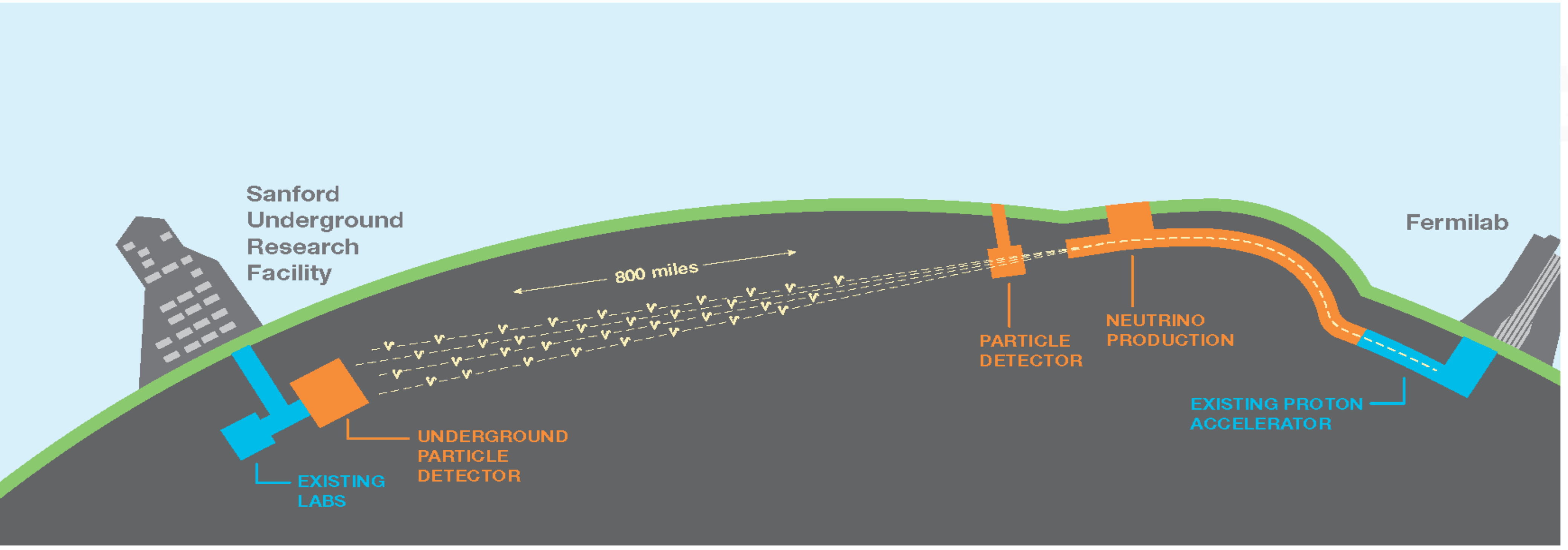

(a)
p
e
νe
Inner Detector
Outer Detector
(b)
(c)
Fiber cable bobbin
multi-PMT
ODPMT

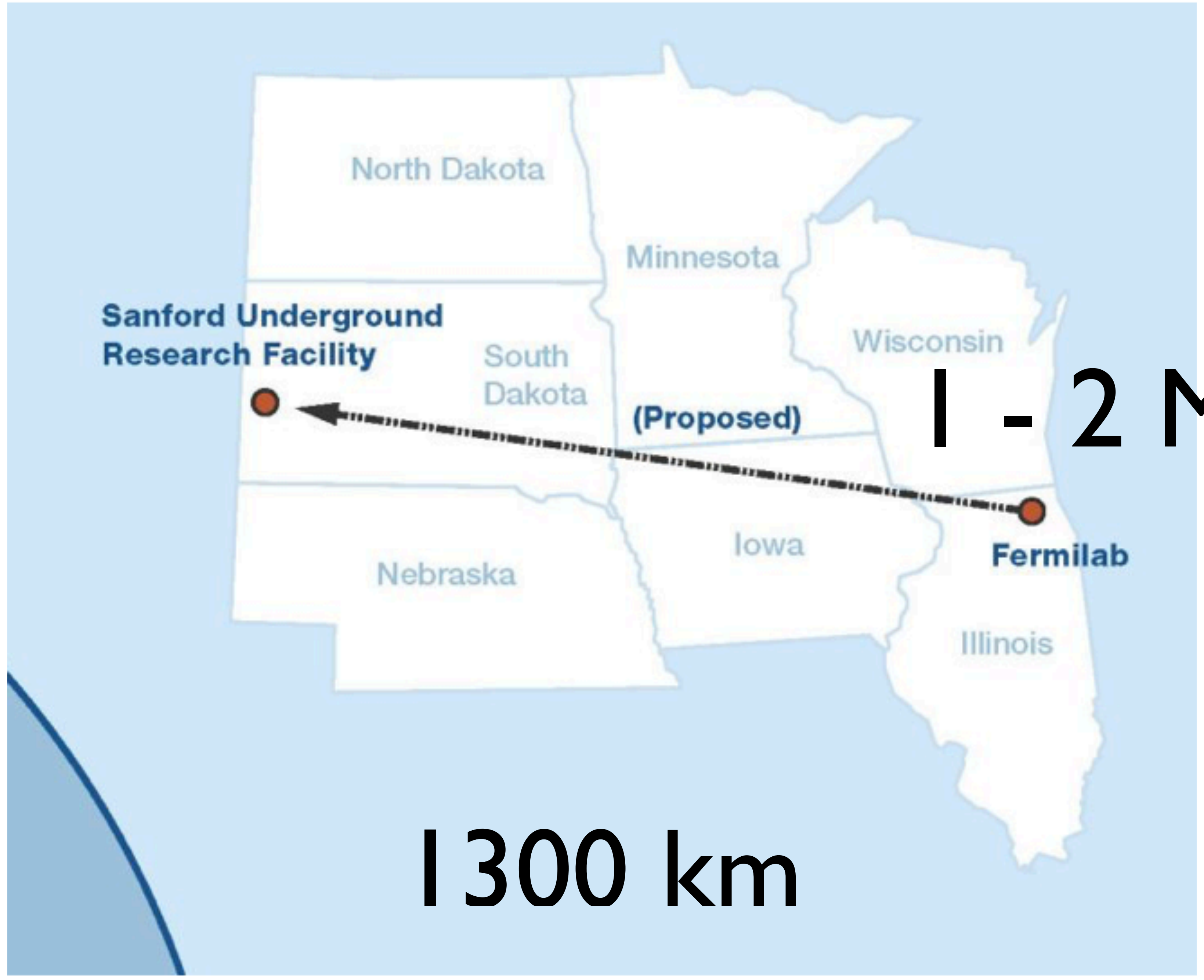
Start 2028

DUNE (70 kt of LAr)

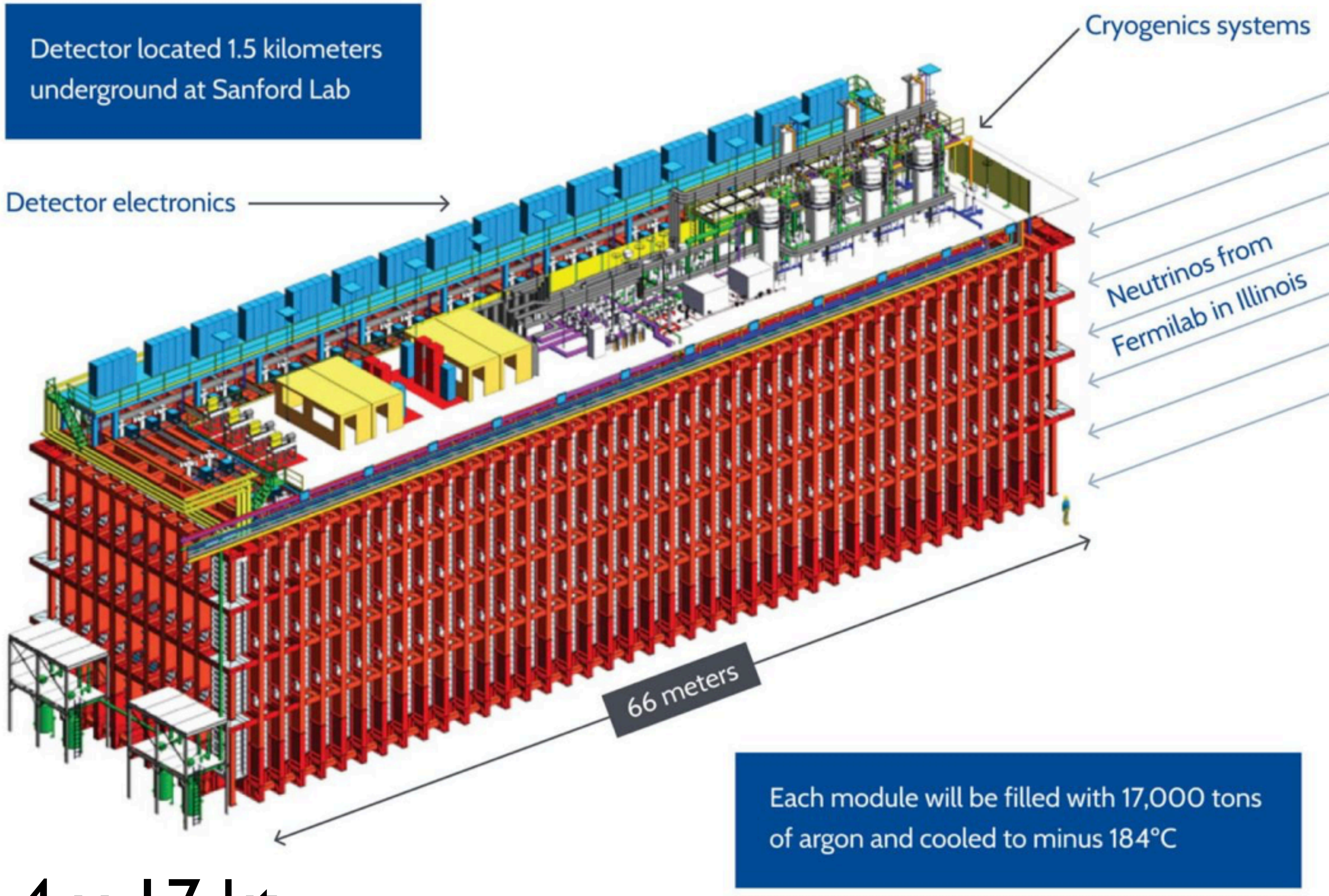
$$\nu_\mu \to \nu_e, \quad \bar{\nu}_\mu \to \bar{\nu}_e$$

$$\nu_\mu \to \nu_\mu, \bar{\nu}_\mu \to \bar{\nu}_\mu$$





1 - 2 MW



4 x 17 kt

Eigenvalues

$$P(\nu_\alpha \to \nu_\beta) = \left| \sum_{j=1}^{3} v_{\alpha j} v^*_{\beta j} \, e^{-i\lambda_j L/(2E)} \right|^2$$

Eigenvectors

(6 nu parameters + 2 exp. Parameters) x 100 bins/parameter

Neutrino Oscillation Exp.

Only BSM physics
Known to be
Accessible !

- Mass Pattern (Ordering)
- Flavor Pattern (Octant)
- CP Violation
- New Physics

} Inter-connected

For Dirac vs Majorana nature other types of experiment are required.

EigenValues of H :

Determination of the Eigenvalues, λ_i

Characteristic eqn.

$$Det[\lambda I - H] = 0 = \lambda^n + d_1 \lambda^{n-1} + \cdots + d_n = \Pi_j(\lambda - \lambda_j)$$

$$d_1 = -Tr[H], \quad d_2 = \frac{1}{2}(Tr^2[H] - Tr[H^2]), \cdots d_n = (-1)^n Det[H]$$

Derivative of the Characteristic eqn.

$$Det'[\lambda I - H] \equiv \frac{\partial}{\partial \lambda} Det[\lambda I - H]$$

$$= n\lambda^{n-1} + (n-1)d_1\lambda^{n-2} + \cdots + d_{n-1}$$

$$= \sum_{k=1}^{n} \Pi_{j \neq k}(\lambda - \lambda_j)$$

$$Det'[\lambda_i I - H] = \Pi_{j \neq i}(\lambda_i - \lambda_j)$$

Example: 3x3

"." means same as above diagonal

$$(2E)\,H = \Delta m^2_{21}\begin{pmatrix} c^2_{13}s^2_{12} & c_{13}s_{12}c_{12} & -s_{13}c_{13}s^2_{12} \\ . & c^2_{12} & -s_{13}s_{12}c_{12} \\ . & . & s^2_{13}s^2_{12} \end{pmatrix}$$

$$+\Delta m^2_{31}\begin{pmatrix} s^2_{13} & 0 & s_{13}c_{13} \\ . & 0 & 0 \\ . & . & c^2_{13} \end{pmatrix} + A_{mat}\begin{pmatrix} 1 & 0 & 0 \\ . & 0 & 0 \\ . & . & 0 \end{pmatrix}$$

$c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$

where $A_{mat} = 2\sqrt{2}G_F N_e E$

$$\lambda^3 - A\lambda^2 + B\lambda - C = 0$$

$$A \equiv -d_1 = \Delta m^2_{31} + \Delta m^2_{21} + A_{mat}$$

$$B \equiv \quad d_2 = \Delta m^2_{31}\Delta m^2_{21} + A_{mat}[\Delta m^2_{31}c^2_{13} + \Delta m^2_{21}(1 - c^2_{13}s^2_{12})]$$

$$C \equiv -d_3 = A_{mat}\,\Delta m^2_{31}\Delta m^2_{21}c^2_{13}c^2_{12}$$

Cardano 1545:

Solar and Atmospheric Resonance Regions

$$\lambda^3 - A\lambda^2 + B\lambda - C = 0$$

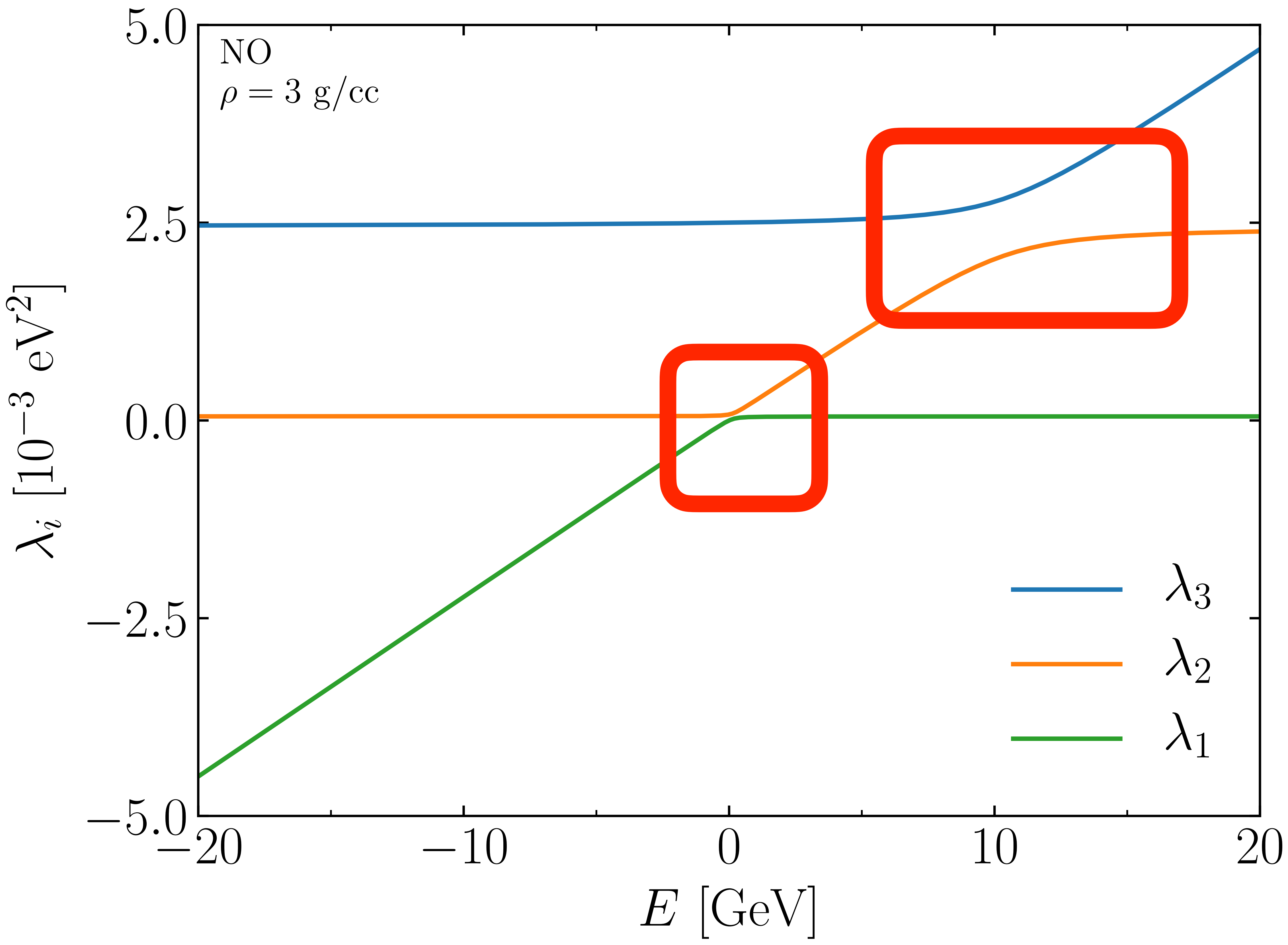
$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2(A^2-3B)^{3/2}}\right]\right\},$$

$$\lambda_1 = \frac{1}{3}A - \frac{1}{3}\sqrt{A^2-3B}\left(S + \sqrt{3}\sqrt{1-S^2}\right),$$

$$\lambda_2 = \frac{1}{3}A - \frac{1}{3}\sqrt{A^2-3B}\left(S - \sqrt{3}\sqrt{1-S^2}\right),$$

$$\lambda_3 = \frac{1}{3}A + \frac{2}{3}\sqrt{A^2-3B}\,S,$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

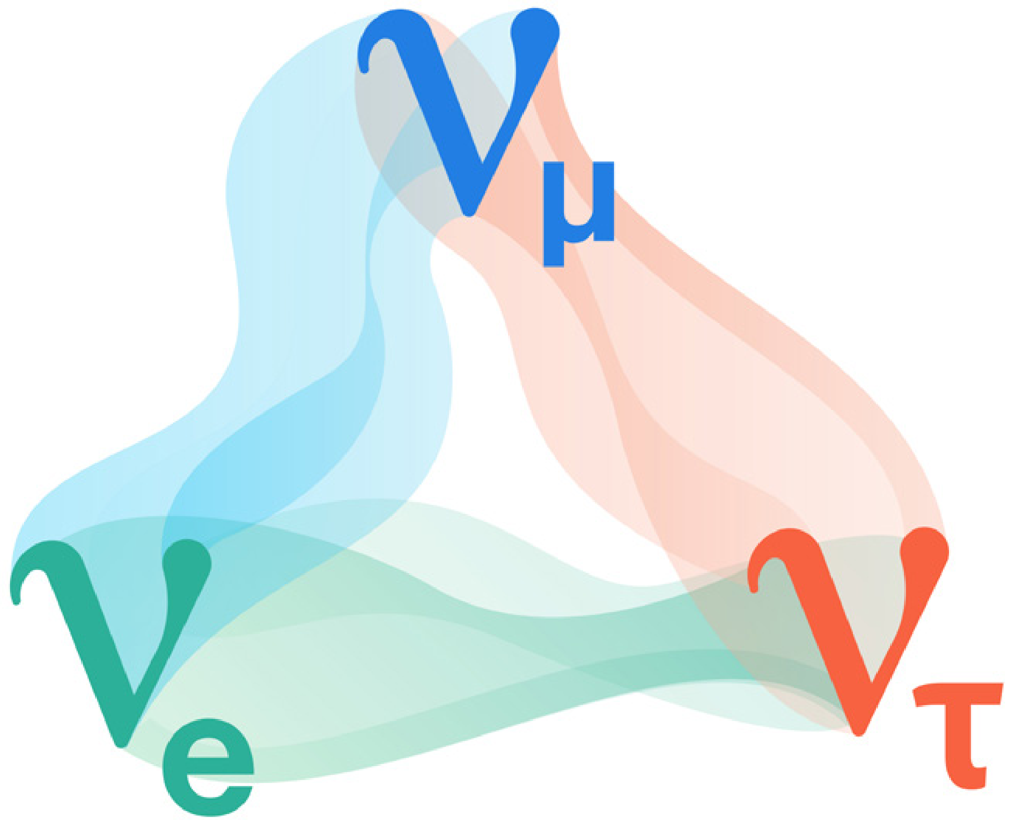


$$A_{mat} = 2\sqrt{2}G_F N_e E$$

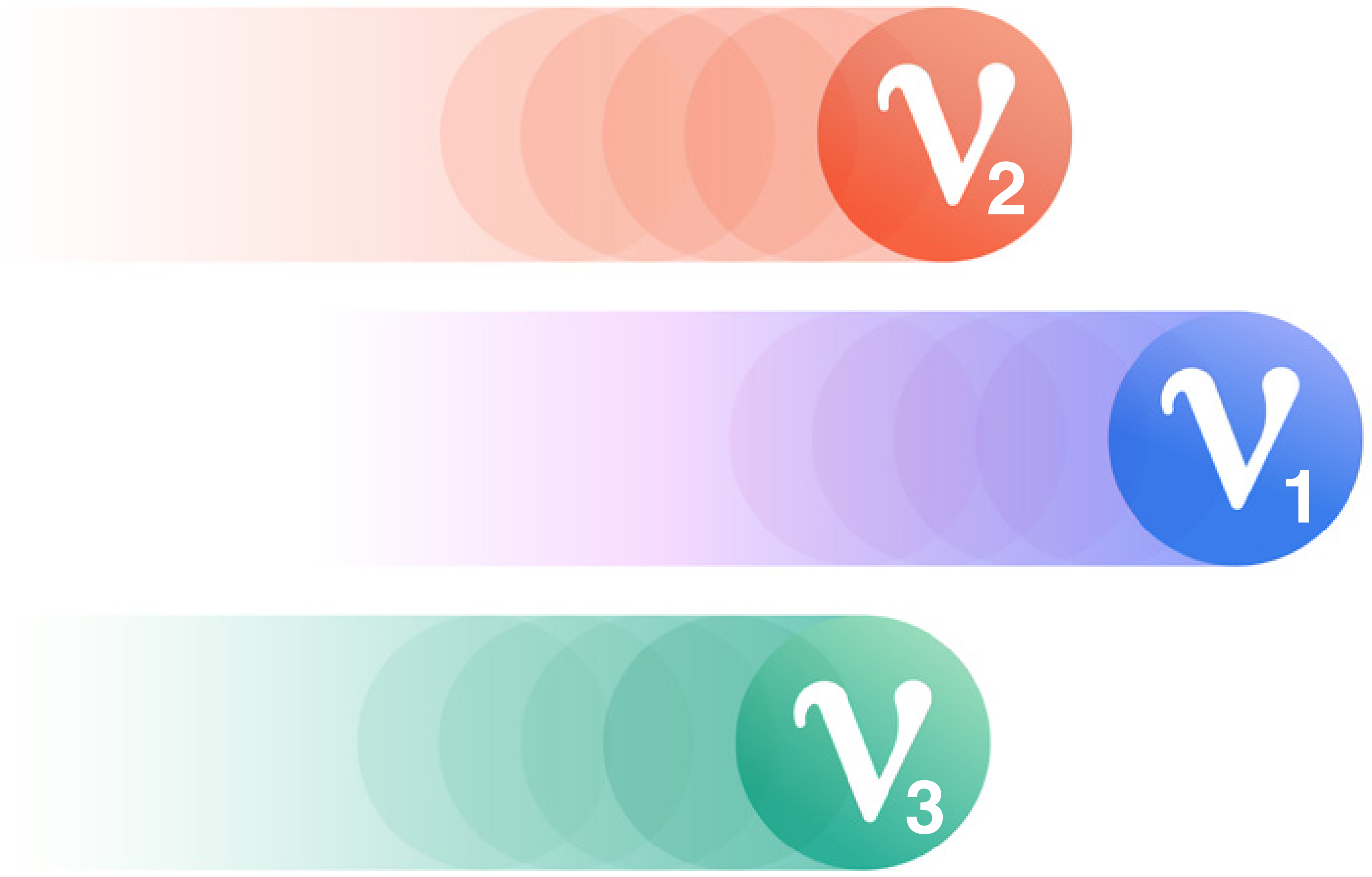
What Physics gives us such a Hamiltonian ?

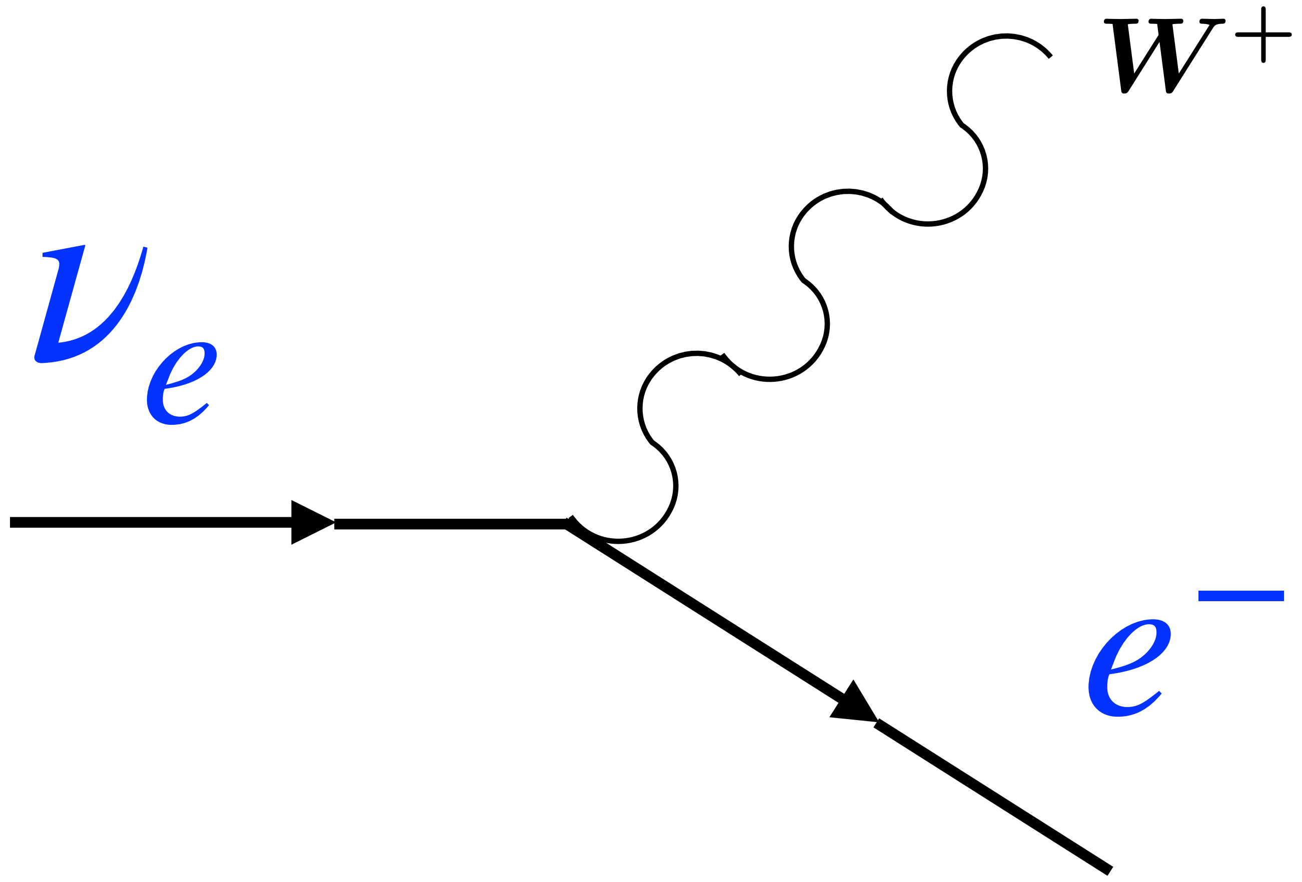
Interactions States :

Propagation / Mass States:



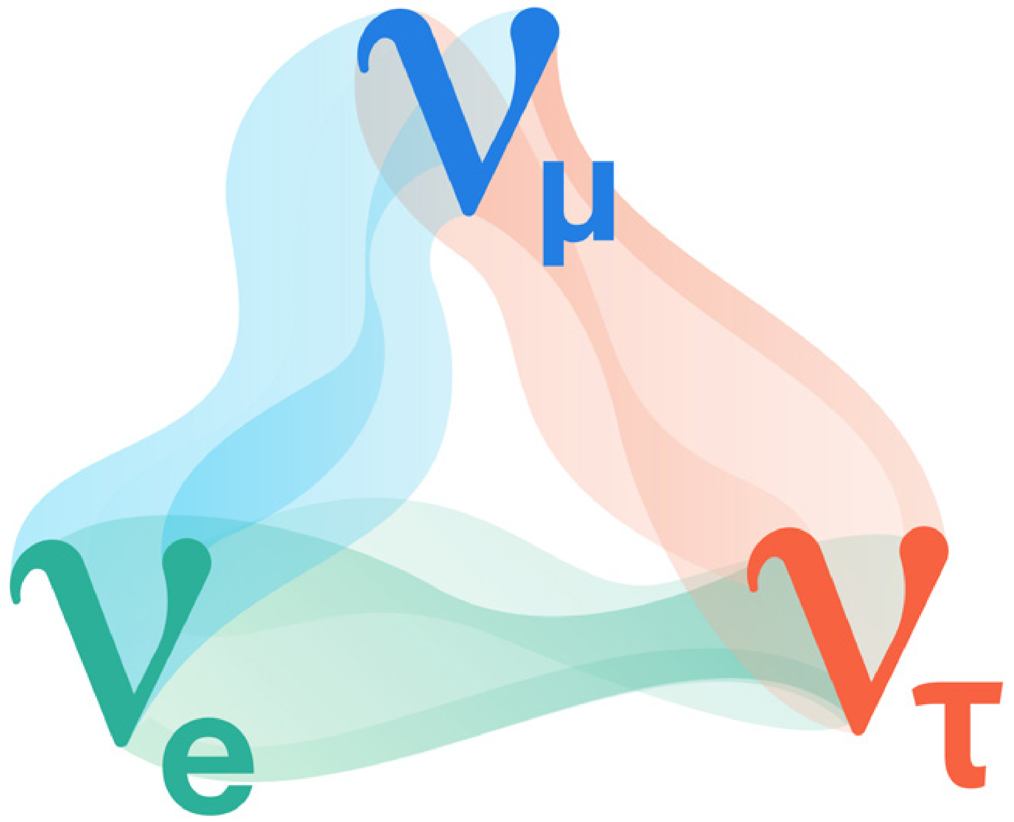
$$= U$$



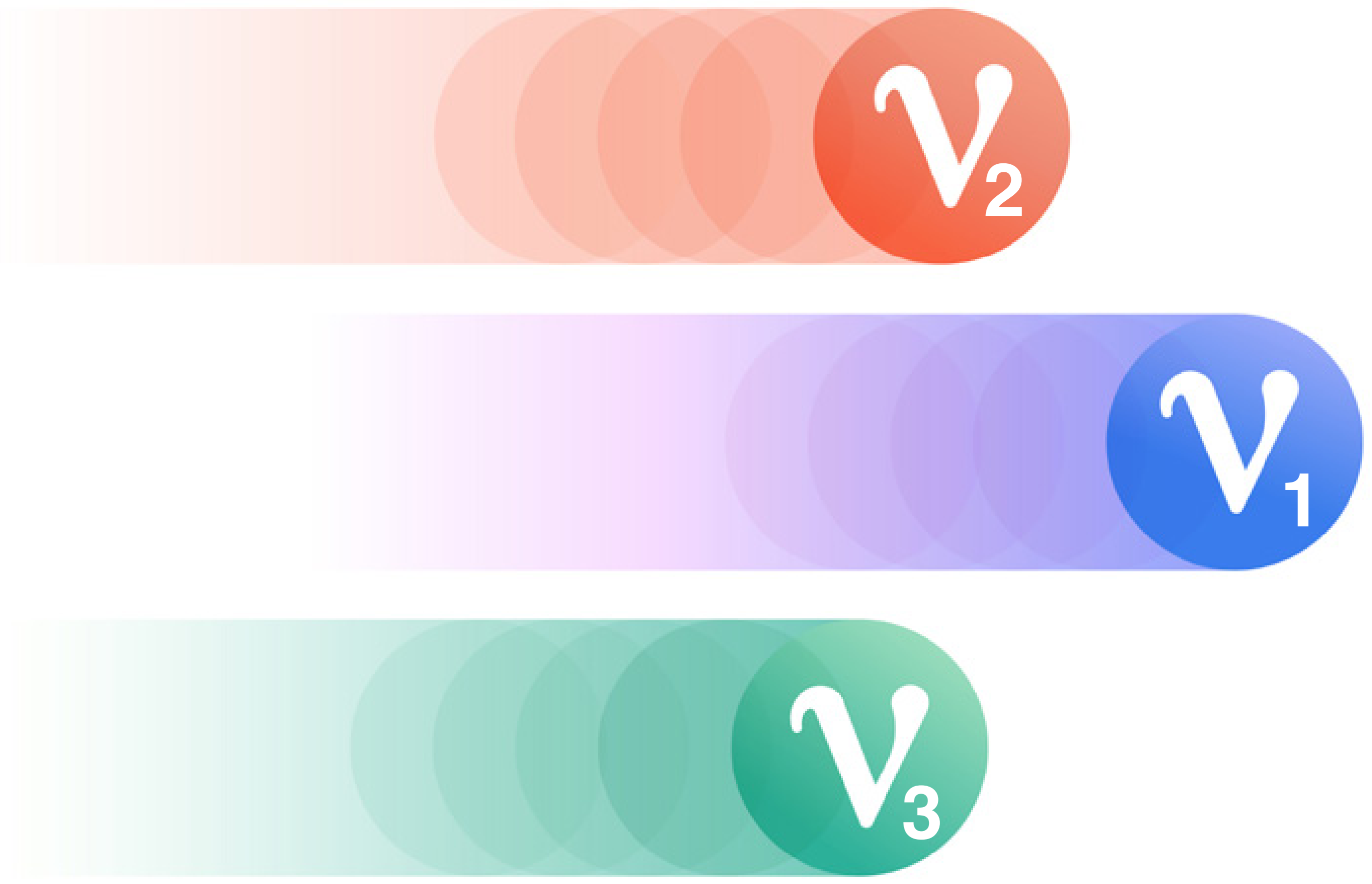


$$e^{-iE_j t}$$

Interactions States : Propagation / Mass States:



$$= U$$



$$U =$$

Complex 3x3 Rotation Matrix:
3 angles $\theta_{12}, \theta_{13}, \theta_{23}$
and
complex phase δ_{CP}
(pdg convention used)

Complex

Neutrino Flavor Basis: $H = H_0 + H_{mat}$

Masses rotated to flavor basis:

$$(2E)H_0 = R_{13}(\theta_{13})R_{12}(\theta_{12}) \begin{pmatrix} 0 & 0 & 0 \\ . & \Delta m^2_{21} & 0 \\ . & . & \Delta m^2_{31} \end{pmatrix} R^T_{12}(\theta_{12})R^T_{13}(\theta_{13})$$

$$R_{13}(\theta_{13}) = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \qquad R_{12}(\theta_{12}) = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$

$$(2E)\,H = \Delta m^2_{21} \begin{pmatrix} c^2_{13}s^2_{12} & c_{13}s_{12}c_{12} & -s_{13}c_{13}s^2_{12} \\ . & c^2_{12} & -s_{13}s_{12}c_{12} \\ . & . & s^2_{13}s^2_{12} \end{pmatrix}$$

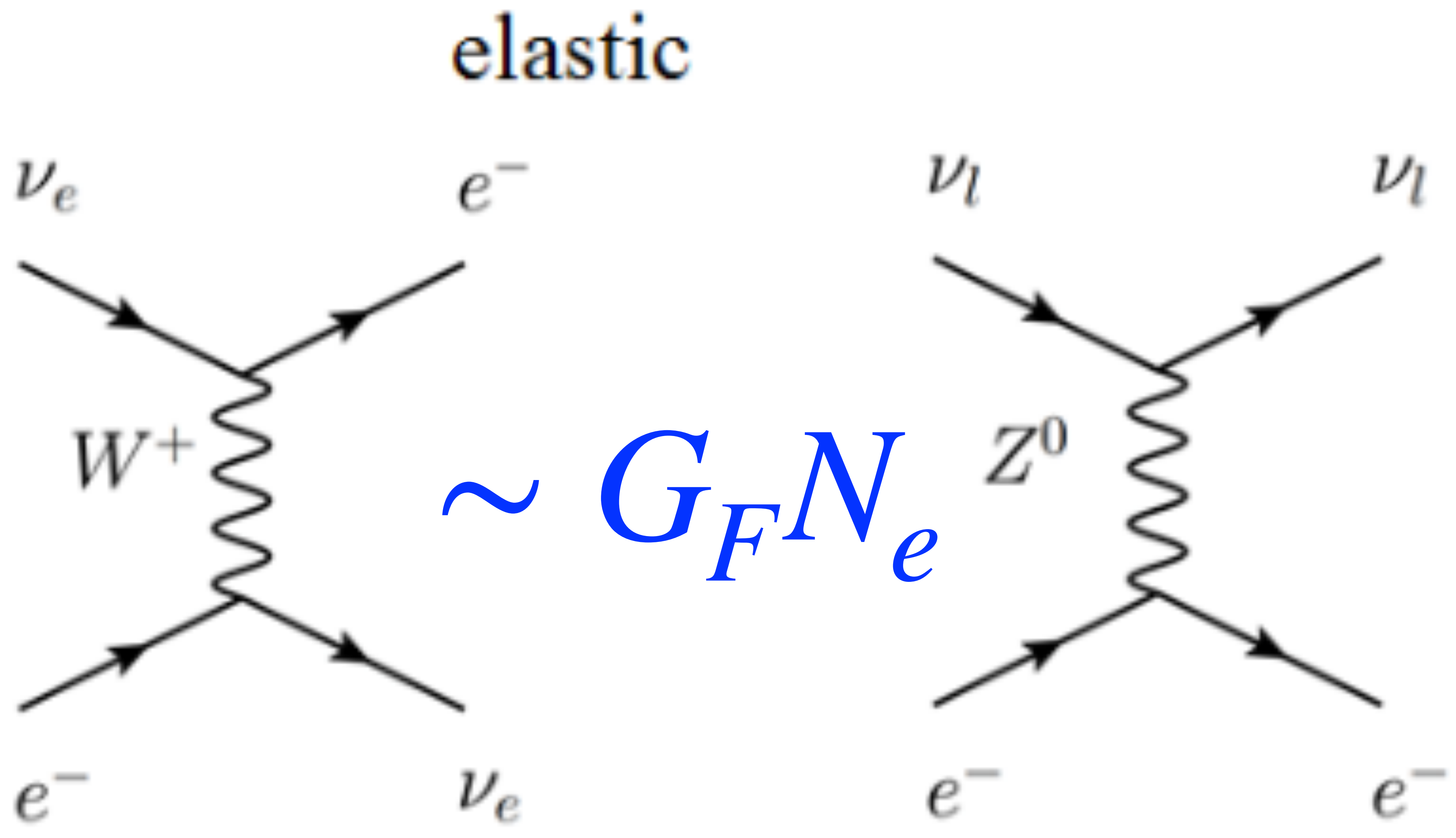
$$+\,\Delta m^2_{31} \begin{pmatrix} s^2_{13} & 0 & s_{13}c_{13} \\ . & 0 & 0 \\ . & . & c^2_{13} \end{pmatrix}$$

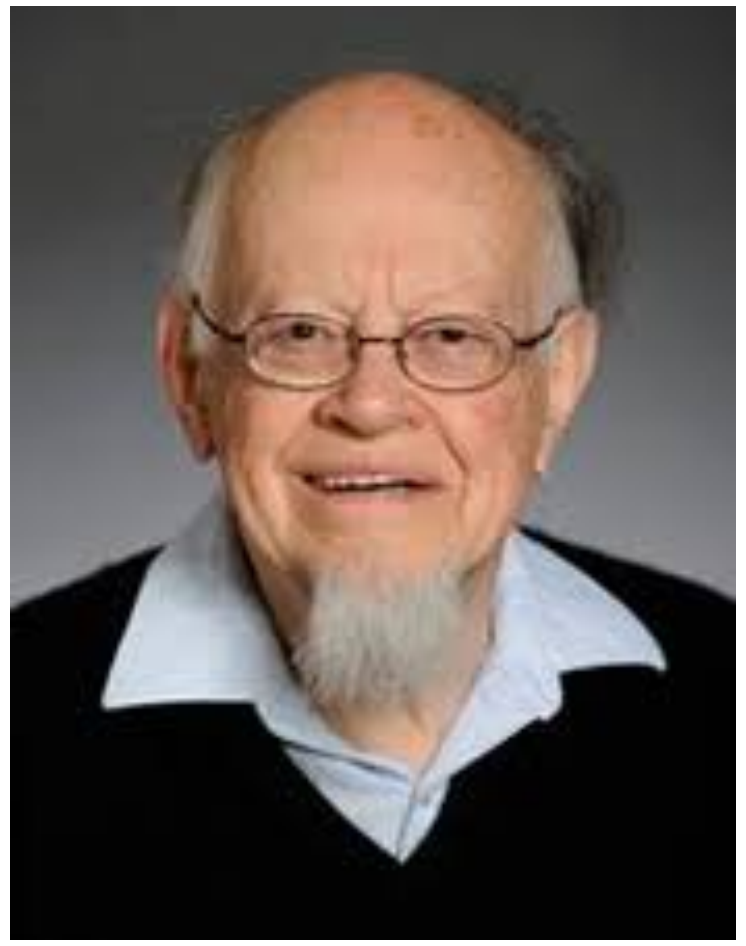
Neutrino Flavor Basis:

ν_e interacting with the electrons in matter:
coherent forward scattering

Lincoln Wolfenstein 1978:
Neutrino Propagation in Matter

$$(2E)\,H_{mat} = A_{mat}\begin{pmatrix} 1 & 0 & 0 \\ . & 0 & 0 \\ . & . & 0 \end{pmatrix}, \quad \text{where} \quad A_{mat} = 2\sqrt{2}G_F N_e E$$





$$(2E)\,H = \Delta m^2_{21} \begin{pmatrix} c^2_{13}s^2_{12} & c_{13}s_{12}c_{12} & -s_{13}c_{13}s^2_{12} \\ . & c^2_{12} & -s_{13}s_{12}c_{12} \\ . & . & s^2_{13}s^2_{12} \end{pmatrix}$$

$$+ \Delta m^2_{31} \begin{pmatrix} s^2_{13} & 0 & s_{13}c_{13} \\ . & 0 & 0 \\ . & . & c^2_{13} \end{pmatrix} + A_{mat} \begin{pmatrix} 1 & 0 & 0 \\ . & 0 & 0 \\ . & . & 0 \end{pmatrix}$$

$\Delta m^2_{21} \approx 7.5 \times 10^{-5}$ eV ^{2}, $\Delta m^2_{31} \approx \pm 2.5 \times 10^{-3}$ eV ^{2} typically: $\Delta m^2_{21} < |A_{mat}| < |\Delta m^2_{31}|$

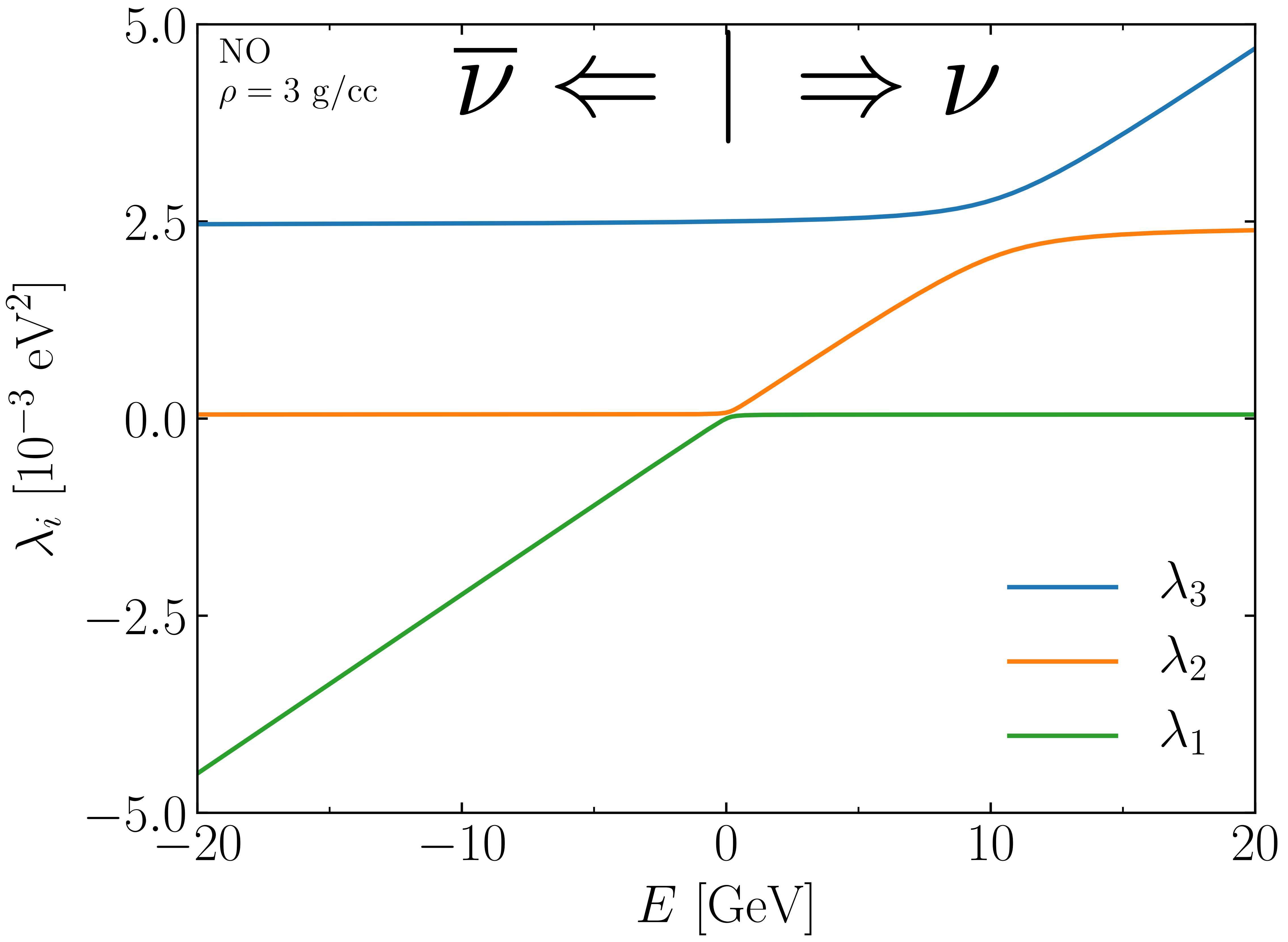
E $\sim$ 1 MeV to 100 GeV
for solar/reactor, accelerator, to atmospheric neutrinos

Cardano 1545:

$$\lambda^3 - A\lambda^2 + B\lambda - C = 0$$

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2(A^2 - 3B)^{3/2}}\right]\right\},$$

$$\lambda_1 = \frac{1}{3}A - \frac{1}{3}\sqrt{A^2 - 3B}\left(S + \sqrt{3}\sqrt{1 - S^2}\right),$$
$$\lambda_2 = \frac{1}{3}A - \frac{1}{3}\sqrt{A^2 - 3B}\left(S - \sqrt{3}\sqrt{1 - S^2}\right),$$
$$\lambda_3 = \frac{1}{3}A + \frac{2}{3}\sqrt{A^2 - 3B}\,S,$$



$$E_i = E + \frac{\lambda_i}{(2E)}$$

EigenVectors of H :

Determination of the Eigenvectors, v_i

$Hv_i = \lambda_i v_i$ rephasing $v_i \Rightarrow e^{i\phi} v_i$ is also eigenvector

$$v_i = \begin{pmatrix} v_{1i} \\ v_{2i} \\ \vdots \\ v_{ni} \end{pmatrix}$$

$$v_{\alpha i} v^*_{\beta i} = \;???$$

chose v_{1i} to be real and positive (or any other phase)

then with $v_{1i}v^*_{1i}, \quad v_{2i}v^*_{1i}, \quad \cdots, \quad v_{ni}v^*_{1i}$

then we have the eigenvector v_i .

only one row or column of $v_{\alpha i} v^*_{\beta i}$ needed !

- Lagrange Method: $v_{\alpha i} v^*_{\beta i} \propto [\Pi_{j\neq i}(H - \lambda_j I)]_{\alpha\beta}$

requires knowing all eigenvalues ! follows because $\Pi_{\text{all j}}(H - \lambda_j I) = 0.$

For normalization divide by: $\Pi_{j\neq i}(\lambda_i - \lambda_j) = Det'[\lambda I - H]|_{\lambda=\lambda_i}$

- Adjugate Method: $v_{\alpha i} v^*_{\beta i} \propto \text{Adjg}[\lambda_i I - H]_{\alpha\beta}$

requires knowing only λ_i eigenvalue !

same normalization factor.

Adjugate of Matrix, X, is the transpose of the Co-factor matrix:

$\text{Adjg}[X]_{ij} \equiv (-1)^{i+j}\text{Det}[\text{sub-matrix of X formed by removing the j-th row and i-th col.}]$

$$X \cdot \text{Adjg}[X] = \text{Adjg}[X] \cdot X = \text{Det}[X] I$$

Physicist's sketch of why this works:

Consider $H = V\text{Diag}[\lambda_1, \lambda_2, \cdots, \lambda_n]V^\dagger$ where the columns of V are the eigenvectors $v_1, v_2, \cdots v_n$.

if $\text{Det}[X] \neq 0$, then $\text{Adjg}[X] = \text{Det}[X]X^{-1}$

$$(\lambda I - H)^{-1} = V\text{Diag}[\tfrac{1}{\lambda-\lambda_1}, \tfrac{1}{\lambda-\lambda_2}, \cdots, \tfrac{1}{\lambda-\lambda_n}]V^\dagger$$

$$\text{Adjg}[\lambda I - H] = \text{Det}[\lambda I - H](\lambda I - H)^{-1}$$

$$\text{Adjg}[\lambda I - H]_{\alpha\beta} = \Pi_j(\lambda - \lambda_j)\sum_k v_{\alpha k}\frac{1}{(\lambda-\lambda_k)}v^*_{\beta k}$$

$$= \Pi_{j\neq i}(\lambda - \lambda_j)\sum_k v_{\alpha k}\frac{(\lambda-\lambda_i)}{(\lambda-\lambda_k)}v^*_{\beta k}$$

$$= \Pi_{j\neq i}(\lambda - \lambda_j)\left[v_{\alpha i}v^*_{\beta i} + \sum_{k\neq i} v_{\alpha k}\frac{(\lambda-\lambda_i)}{(\lambda-\lambda_k)}v^*_{\beta k}\right]$$

0

take the limit $\lambda \to \lambda_i$:

$$v_{\alpha i}v^*_{\beta i} = \frac{\text{Adjg}[\lambda_i I - H]_{\alpha\beta}}{\Pi_{j\neq i}(\lambda_i - \lambda_j)}$$

Eigenvector-Eigenvalue Identity:

Diagonal elements of $v_{\alpha i} v^*_{\beta i}$

Characteristic Eqn for H_α evaluated at λ_i

$$|v_{\alpha i}|^2 = \frac{\text{Adjg}[\lambda_i I_n - H]_{\alpha\alpha}}{\Pi_{j\neq i}(\lambda_i - \lambda_j)} = \frac{\text{Det}[\lambda_i I_{n-1} - H_\alpha]}{\Pi_{j\neq i}(\lambda_i - \lambda_j)} = \frac{\Pi_k(\lambda_i(H) - \lambda_k(H_\alpha))}{\Pi_{k\neq i}(\lambda_i(H) - \lambda_k(H))}$$

$$\lambda I - H_4 = \begin{pmatrix} (\lambda - H_{11}) & -H_{12} & -H_{13} & -H_{14} & -H_{15} \\ -H_{21} & (\lambda - H_{22}) & -H_{23} & -H_{24} & -H_{25} \\ -H_{31} & -H_{32} & (\lambda - H_{33}) & -H_{34} & -H_{35} \\ -H_{41} & -H_{42} & -H_{43} & (\lambda - H_{44}) & -H_{45} \\ -H_{51} & -H_{52} & -H_{53} & -H_{54} & (\lambda - H_{55}) \end{pmatrix}$$

Does NOT work for off-diagonal elements !

To obtain the phases one could rotate the original matrix and perform the analysis again !

Eigenvector-eigenvalue identity [edit]

For a Hermitian matrix A, the norm squared of the α-th component of a normalized eigenvector can be calculated using only the matrix eigenvalues and the eigenvalues of the corresponding minor matrix,

$$|v_{i\alpha}|^2 = \frac{\prod_k (\lambda_i(A) - \lambda_k(A_\alpha))}{\prod_{k \neq i} (\lambda_i(A) - \lambda_k(A))},$$

where A_α is the submatrix formed by removing the α-th row and column from the original matrix.[33][34][35] This identity also extends to diagonalizable matrices, and has been rediscovered many times in the literature.[34][36]

Why is the RHS positive and ≤ 1 ?

Cauchy Interlacing Inequalities:

Ordering the Eigenvalues for both A and A_α then its known

$$\lambda_j(A) < \lambda_j(A_\alpha) < \lambda_{j+1}(A)$$

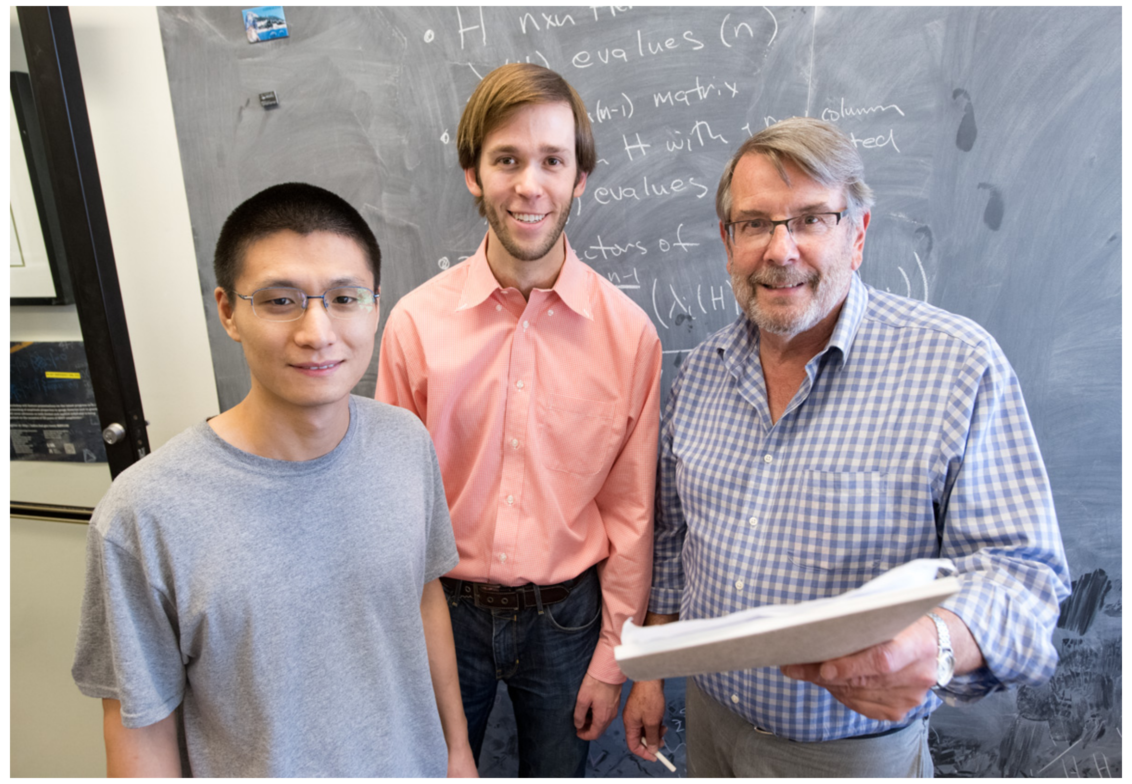
the (n-1) Eigenvalues of A_α are between the n Eigenvalues of A.

$x = \lambda_j(A)$ x-o—x—o-x—o—x $o = \lambda_j(A_\alpha)$

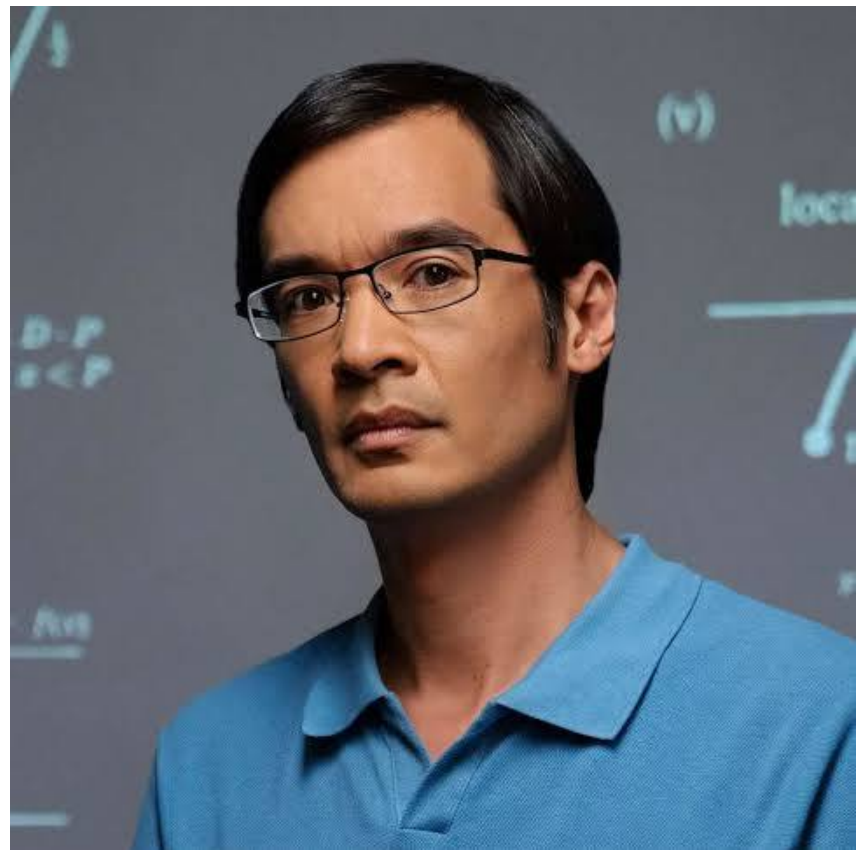
Quanta magazine

MATHEMATICAL PHYSICS

Neutrinos Lead to Unexpected Discovery in Basic Math



Xining Zhang, Peter Denton
+SP



Terence Tao

ALL EPISODES >

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APPLE SPOTIFY

arXiv:1908.03795

arXiv:1908.03795v4 [math.RA]

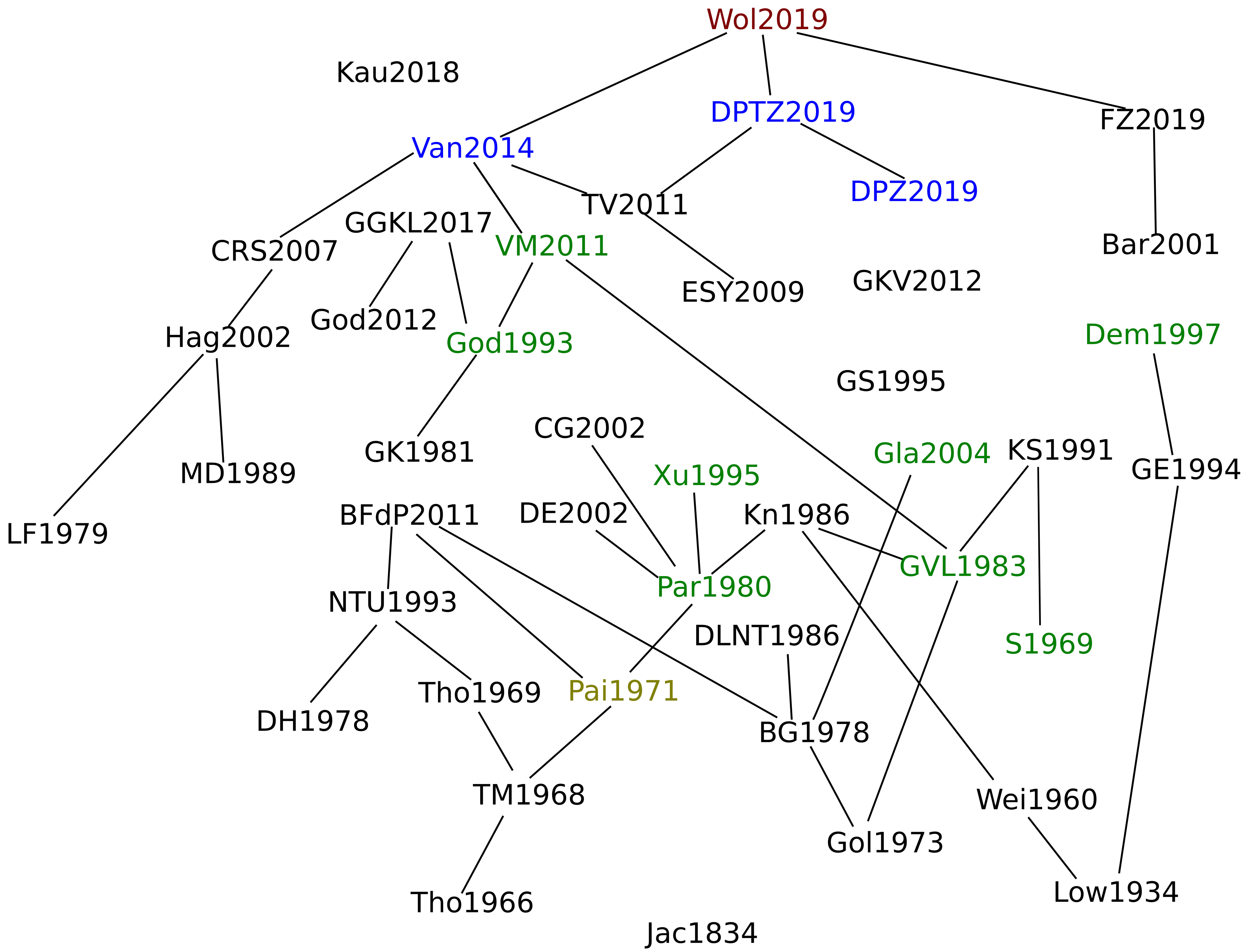


FIGURE 1. The citation graph of all the references in the literature we are aware of (predating the current survey) that mention some variant of the eigenvector-eigenvalue identity. To reduce clutter, transitive references (e.g., a citation of a paper already cited by another paper in the bibliography) are omitted. Note the very weakly connected nature of the graph, with many early initial references not being (transitively) cited by many of the more recent references. Blue references are preprints, green references are books, the brown reference is a thesis, and the red reference is a popular science article. This graph was mostly crowdsourced from feedback received by the authors after the publication of [Wol2019]. The reference [Jac1834] predates all others found by a century!

[Jac1834] C. G. J. Jacobi. De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas tranformandis, quae solis quadratis variabilium constant; una cum variis theorematis de tranformatione etdeterminatione integralium multiplicium. *J. Reine Angew. Math.*, 12:1–69, 1834.

Iterative Algorithm by Le Verrier - Faddeev

(aside)

Le Verrier-Faddeev Algorithm

$$\text{Det}(\lambda I - H) = \lambda^n + \lambda^{n-1} d_1 + \lambda^{n-2} d_2 + \cdots + d_n\,,$$
$$\text{Adjg}(\lambda I - H) \equiv \lambda^{n-1} A_1 + \lambda^{n-2} A_2 + \cdots + A_n\,,$$

where for nxn matrix: $d_n = (-1)^n \text{Det}[H]$ and $A_n = (-1)^{n-1}\text{Adjg}[H]$.

Start with $A_1 = I$, then $d_j = -\frac{1}{j} Tr[H A_j]$ then $A_{j+1} = H A_j + d_j I$

$$d_1 = -Tr[H], \qquad A_2 = H + d_1 I = H - Tr[H] I$$
$$d_2 = \frac{1}{2}(Tr^2[H] - Tr[H^2]), \quad A_3 = H^2 - HTr[H] + d_2 I$$
$$d_3 = \frac{1}{6}(Tr^3[H] - 3Tr[H]Tr[H^2] + 2Tr[H^3]), \quad \cdots$$

For an nxn matrix, one can show that

$$A_{n+1} = H A_n + d_n I = (-1)^{n-1}\Big(H\text{Adjg}[H] - \text{Det}[H] I \Big) = 0$$

thus terminating the series. Good cross check !

3 flavor Neutrino Example:

$$v_{\alpha i}v^*_{\beta i} = \frac{(\ \lambda_i^2 I \ + \ \lambda_i\, (2E)(H - Tr[H]I) \ + \ \text{Adjg}[(2E)H]\ )_{\alpha\beta}}{\Pi_{j\neq i}(\lambda_i - \lambda_j)}$$

Only the first row is needed: $|v_{ei}|^2, \ \ v_{ei}v^*_{\mu i}, \ \ v_{ei}v^*_{\tau i}$

1st row: $(2E)(\text{Tr}[H]I - H) = \Delta m^2_{21}\,(\ 1 - c^2_{13}s^2_{12} \quad -c_{13}s_{12}c_{12} \quad s_{13}c_{13}s^2_{12}\ )$
$+ \Delta m^2_{31}\,(\ c^2_{13} \quad 0 \quad -s_{13}c_{13}\ )$

1st row: $\text{Adj}[(2E)H] = \Delta m^2_{31}\Delta m^2_{21}\,(\ c^2_{13}c^2_{12} \quad -c_{13}s_{12}c_{12} \quad -s_{13}c_{13}c^2_{12}\ )$

3 flavor Neutrino Example:

arXiv: 2511.04735

$$(2E)(\mathrm{Tr}[H]I - H) = \Delta m^2_{21} \begin{pmatrix} 1 - c^2_{13}s^2_{12} & -c_{13}s_{12}c_{12} & s_{13}c_{13}s^2_{12} \\ . & s^2_{12} & s_{13}s_{12}c_{12} \\ . & . & 1 - s^2_{13}s^2_{12} \end{pmatrix} + \Delta m^2_{31} \begin{pmatrix} c^2_{13} & 0 & -s_{13}c_{13} \\ . & 1 & 0 \\ . & . & s^2_{13} \end{pmatrix} + A_{mat} \begin{pmatrix} 0 & 0 & 0 \\ . & 1 & 0 \\ . & . & 1 \end{pmatrix} .$$

$$\mathrm{Adj}[(2E)H] = \Delta m^2_{31}\Delta m^2_{21} \begin{pmatrix} c^2_{13}c^2_{12} & -c_{13}s_{12}c_{12} & -s_{13}c_{13}c^2_{12} \\ . & s^2_{12} & s_{13}s_{12}c_{12} \\ . & . & s^2_{13}c^2_{12} \end{pmatrix} + A_{mat}\,\Delta m^2_{21} \begin{pmatrix} 0 & 0 & 0 \\ . & s^2_{13}s^2_{12} & s_{13}s_{12}c_{12} \\ . & . & c^2_{12} \end{pmatrix} + A_{mat}\,\Delta m^2_{31} \begin{pmatrix} 0 & 0 & 0 \\ . & c^2_{13} & 0 \\ . & . & 0 \end{pmatrix} .$$

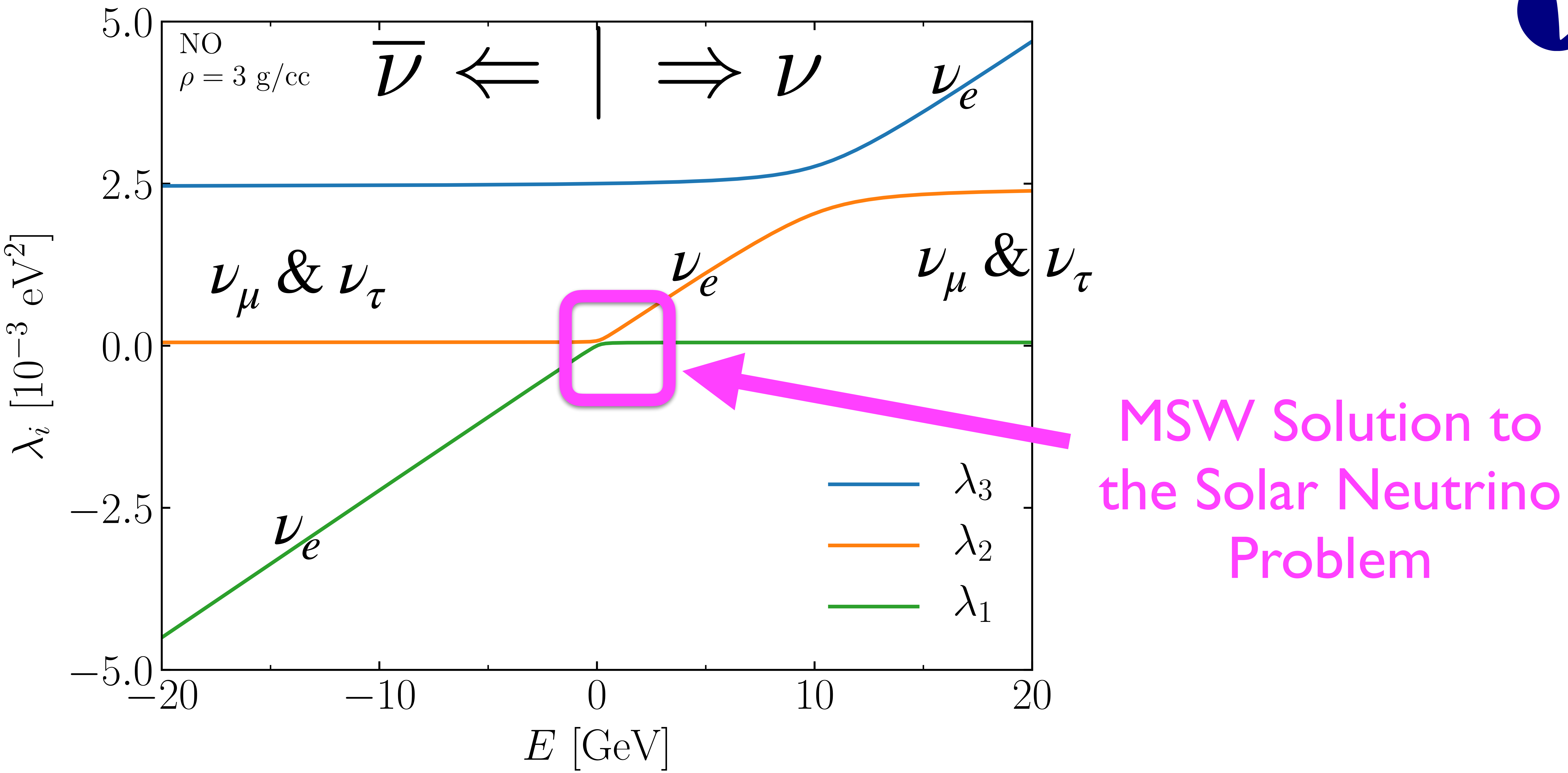
Many cross checks possible:

$$\frac{|v_{\alpha i}|^2 |v_{\beta i}|^2}{|v_{\alpha i} v^*_{\beta i}|^2} = 1, \quad \alpha \neq \beta$$

(normalization factors cancel)

Numerator is 4th order polynomial in λ_i,
whereas Denominator is only 2nd order.

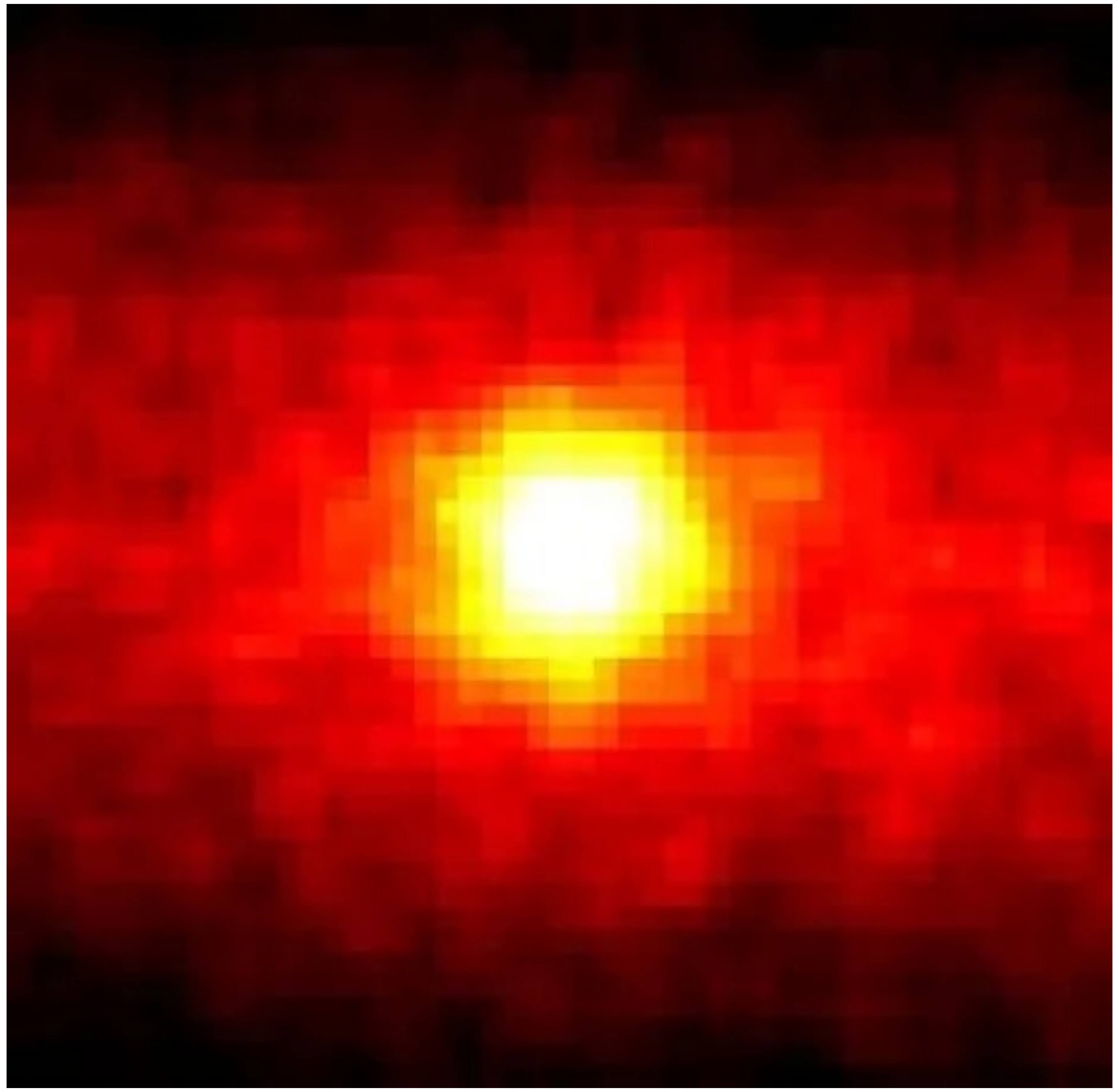
Need to use characteristic equation, for λ_i,
to reduce order of numerator !



$$E_i = E + \frac{\lambda_i}{(2E)}$$

Image of the the Sun using Neutrinos: SuperK

$E_\nu > 8$ MeV

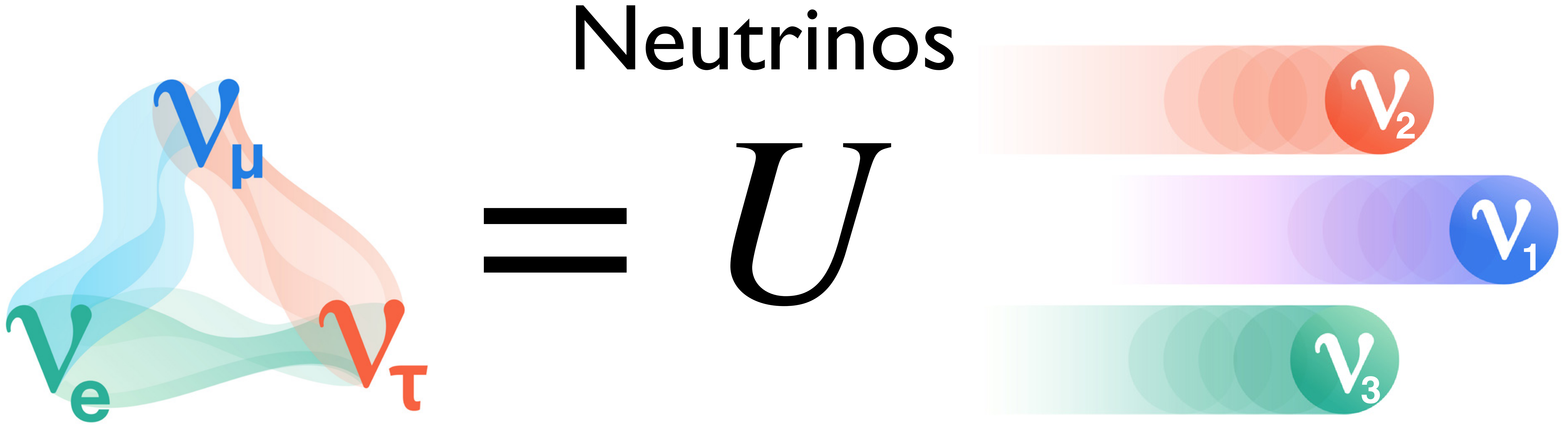


$(\nu_e, \nu_\mu, \nu_\tau) \sim (76, 12, 12)$ %

$(\nu_1, \nu_2, \nu_3) \sim (8, 90, 2)$ %

Dominated by ν_2 neutrinos !!!

CP Violation is encoded in the complex nature of U



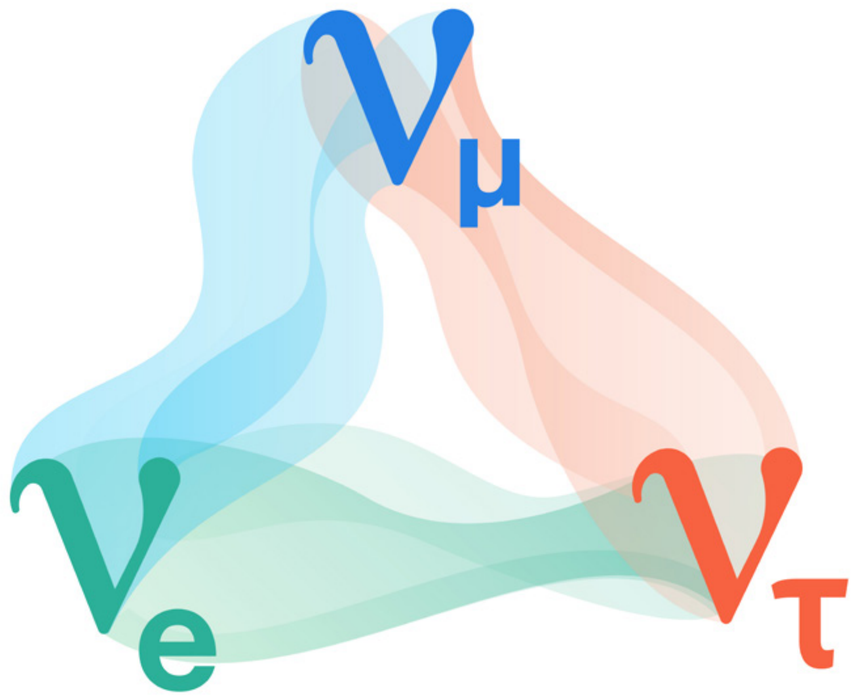
$$U = U_{23}(\theta_{23}, \delta) R_{13}(\theta_{13}) R_{12}(\theta_{12})$$

$$= \begin{pmatrix} 1 & & \\ & c_{23} & s_{23}e^{+i\delta} \\ & -s_{23}e^{-i\delta} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13} \\ & 1 & \\ -s_{13} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

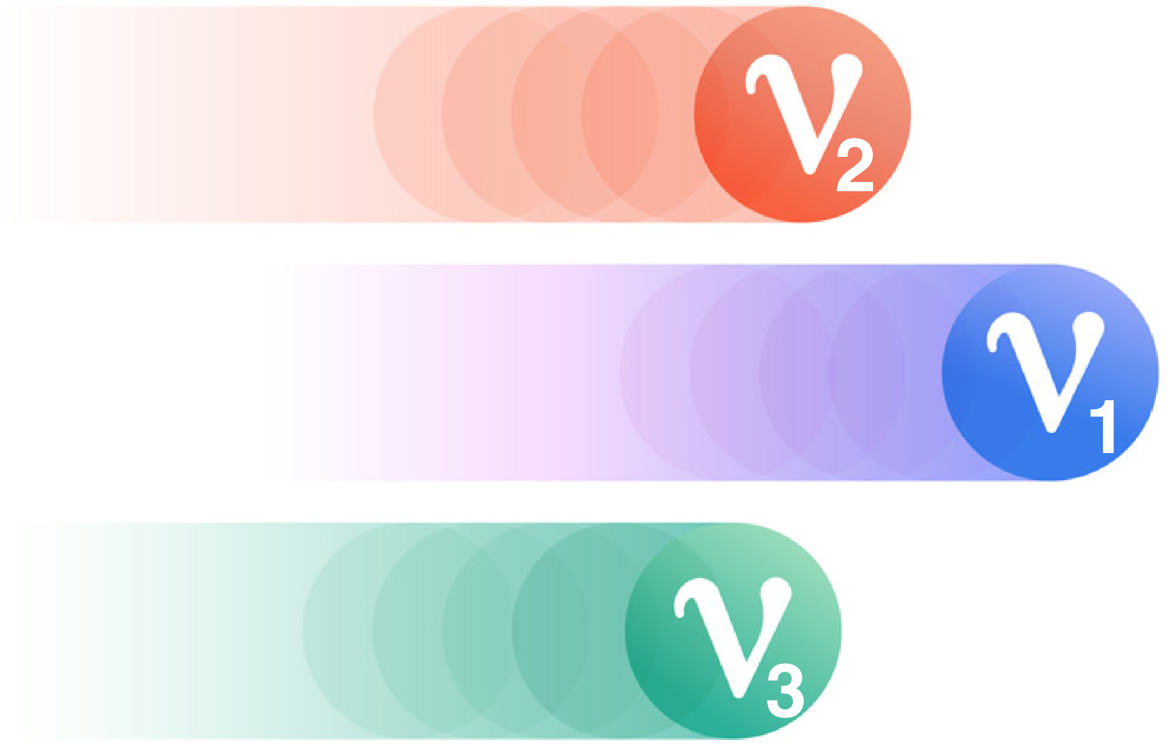
$$U_{23}(\theta_{23}, \delta)\; H_{mat}\; U^\dagger_{23}(\theta_{23}, \delta) = H_{mat}$$

CP Violation is encoded in the complex nature of U

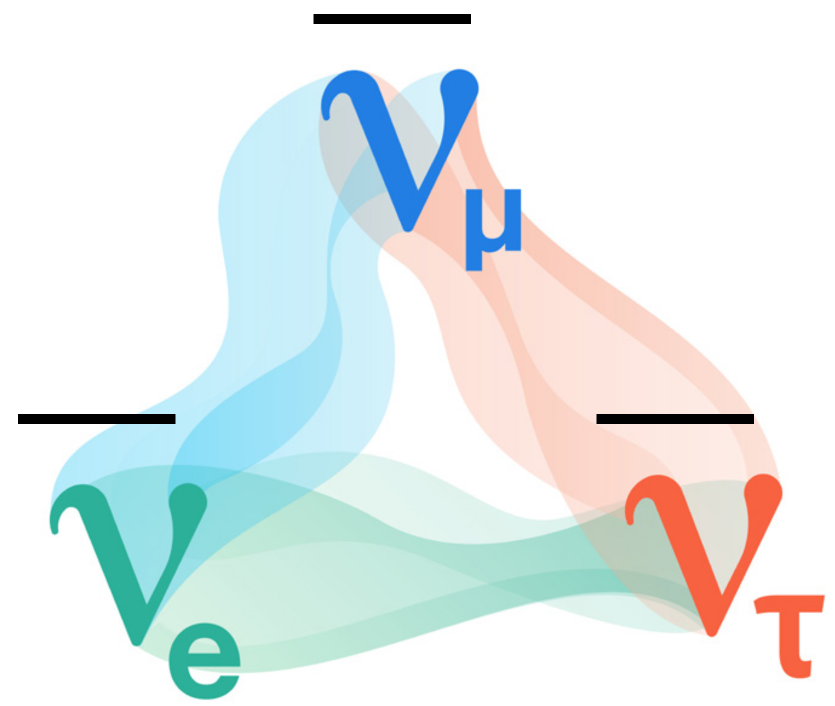
Neutrinos



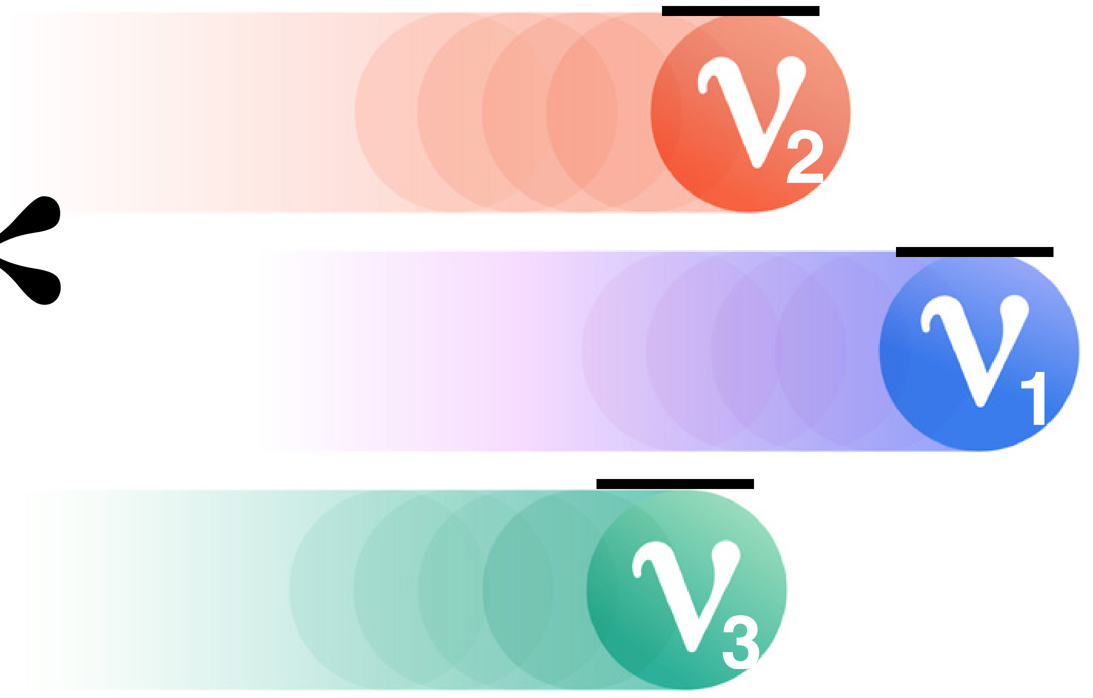
$= U$



Anti-Neutrinos



$= U^*$



$U =$

Complex 3x3 Rotation Matrix:

3 angles $\theta_{12}, \theta_{13}, \theta_{23}$

and

complex phase δ_{CP}

The 2023 **EPS High Energy and Particle Physics Prize** is awarded to

Cecilia Jarlskog for the discovery of an invariant measure of CP violation in both quark and lepton sectors; and …

Jarlskog Invariant: 1985

$$J_{ij}^{\alpha\beta} \equiv \Im\{U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\} = J \sum_{k,\gamma} \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} = 0, \pm 1$$

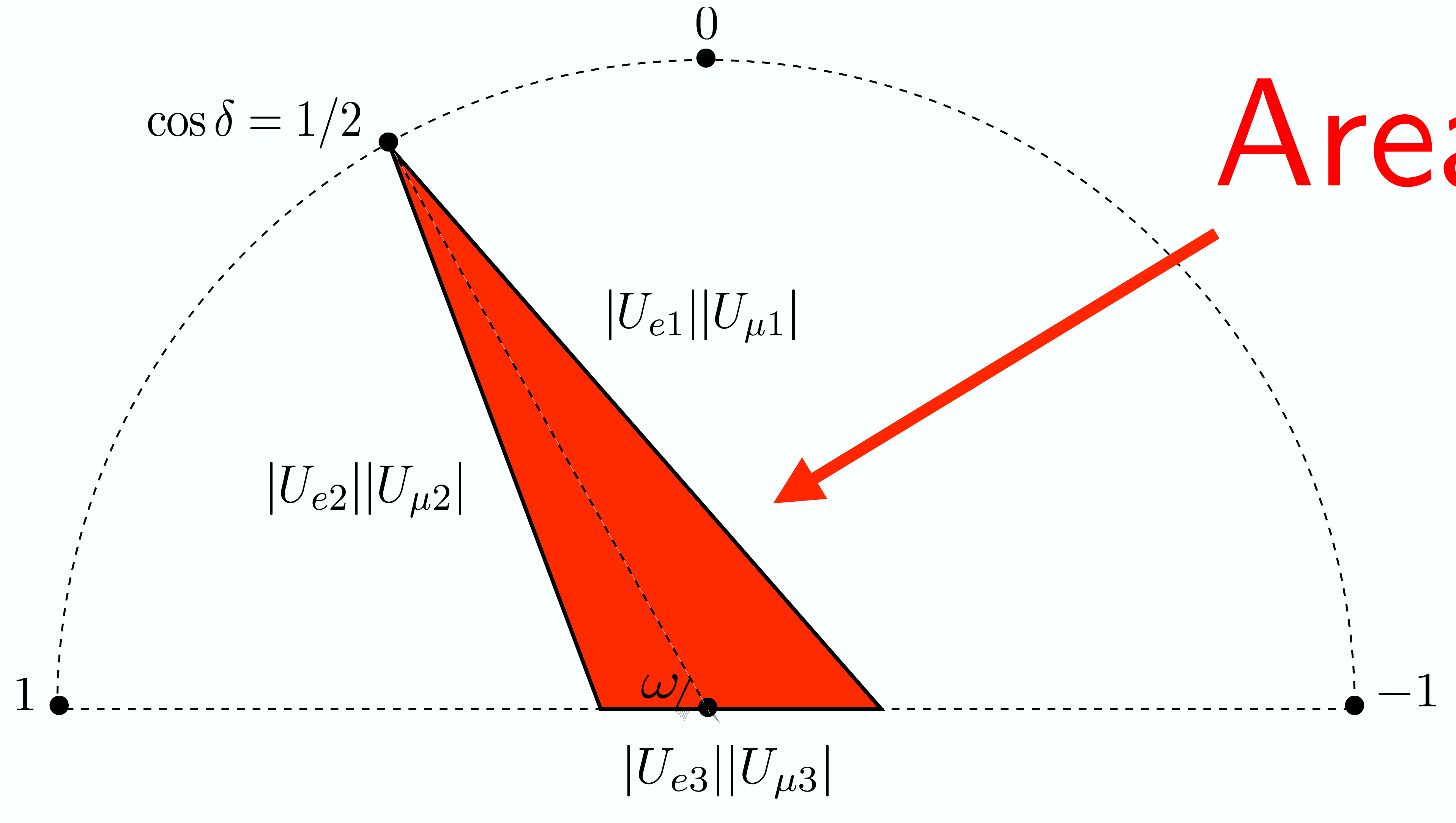
$$J_{pdg} = s_{23}c_{23}\ s_{13}c_{13}^2\ s_{12}c_{12}\ \sin\delta$$

$$J_l = (3.36 \pm 0.06) \sin\delta_{CP} \times 10^{-2}$$

Quarks

$$J = (3.08 \pm 0.14) \times 10^{-5}$$

also used in SMEFT



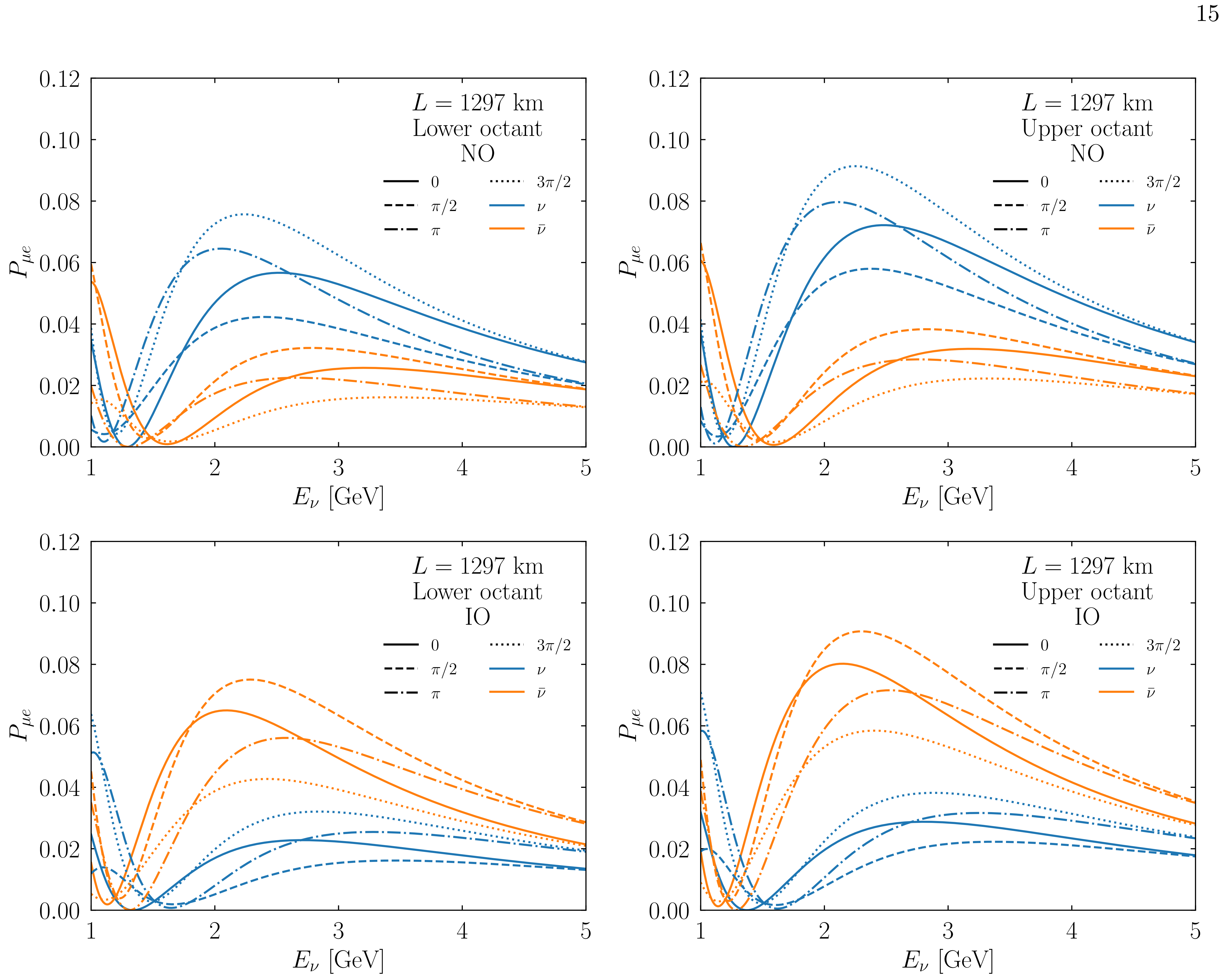
Area $= \frac{1}{2}\,|J|$

To include CP, we need to replace $v_i \Rightarrow U_{23}(\theta_{23}, \delta) v_i$

$$P(\nu_\alpha \to \nu_\beta) = \left| \sum_j v_{\alpha j} v^*_{\beta j} \, e^{-i\lambda_j L/(2E)} \right|^2$$

DUNE oscillation probabilities :

$P_{\mu e}$

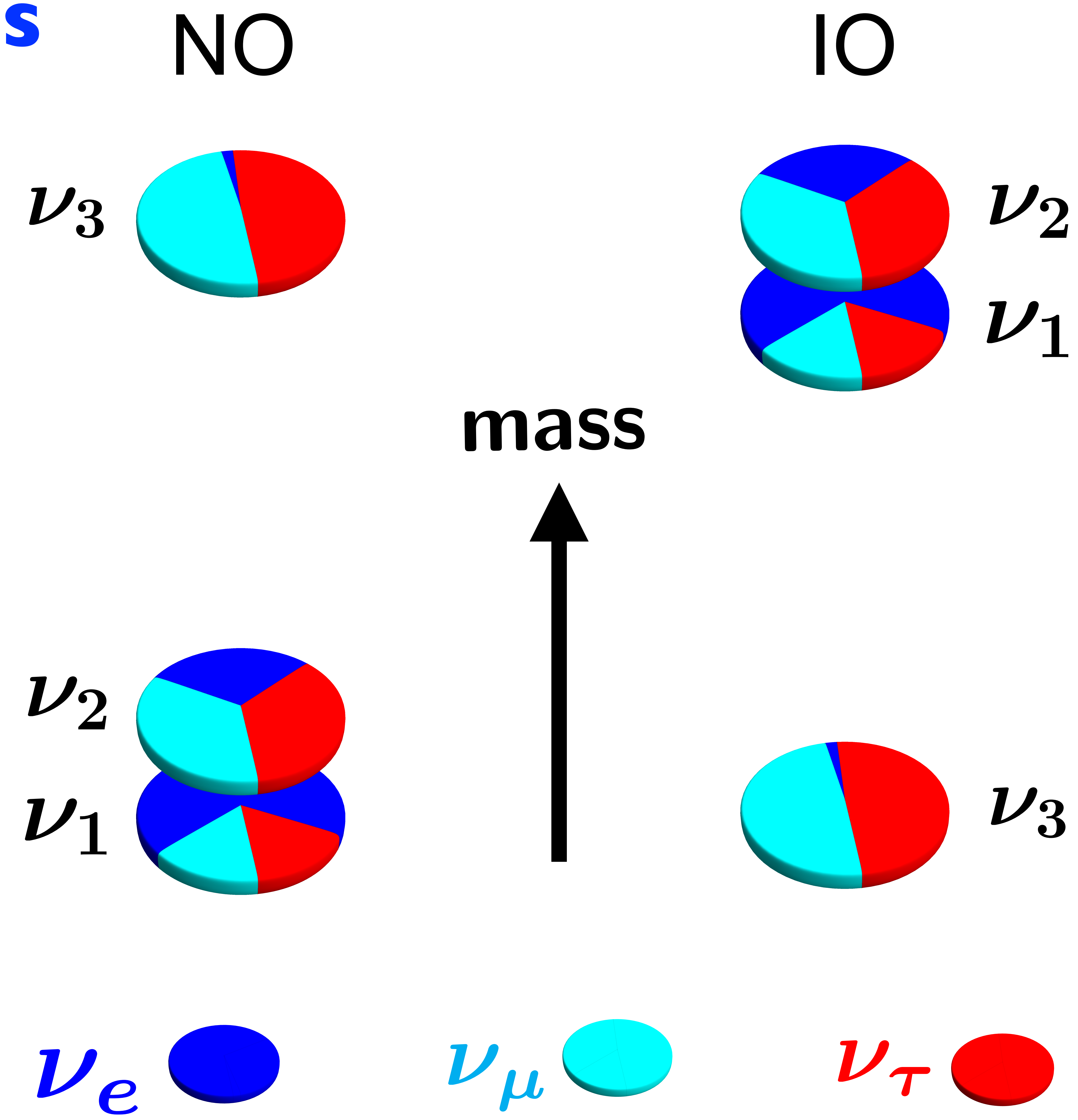


NuFast-LBL arXiv:2405.02400
takes 2 x Vacuum code

NuFast-Earth arXiv:2511.04735
Order of magnitude fast than other codes

Neutrino Mass Ordering

Normal Ordering



Inverted Ordering

$$|U_{\mu 3}|^2 = \cos^2\theta_{13}\sin^2\theta_{23}$$

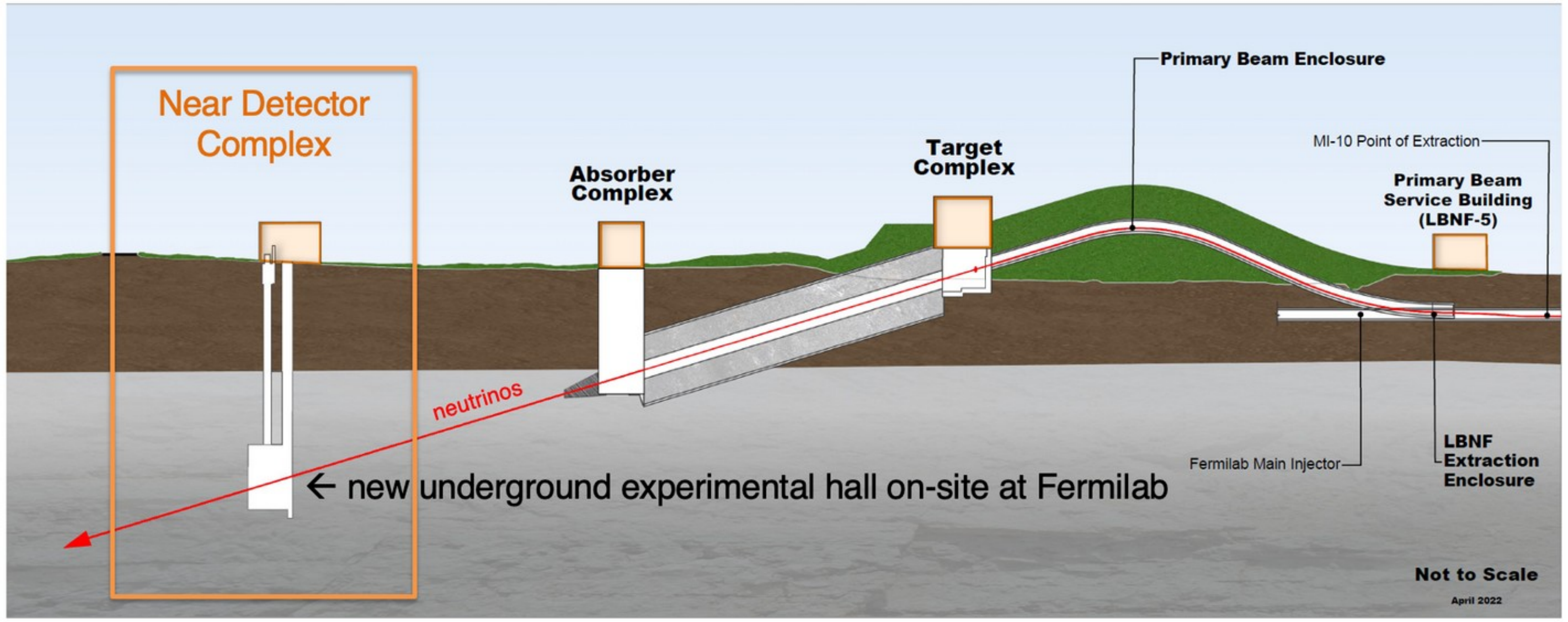
$$|U_{\tau 3}|^2 = \cos^2\theta_{13}\cos^2\theta_{23}$$

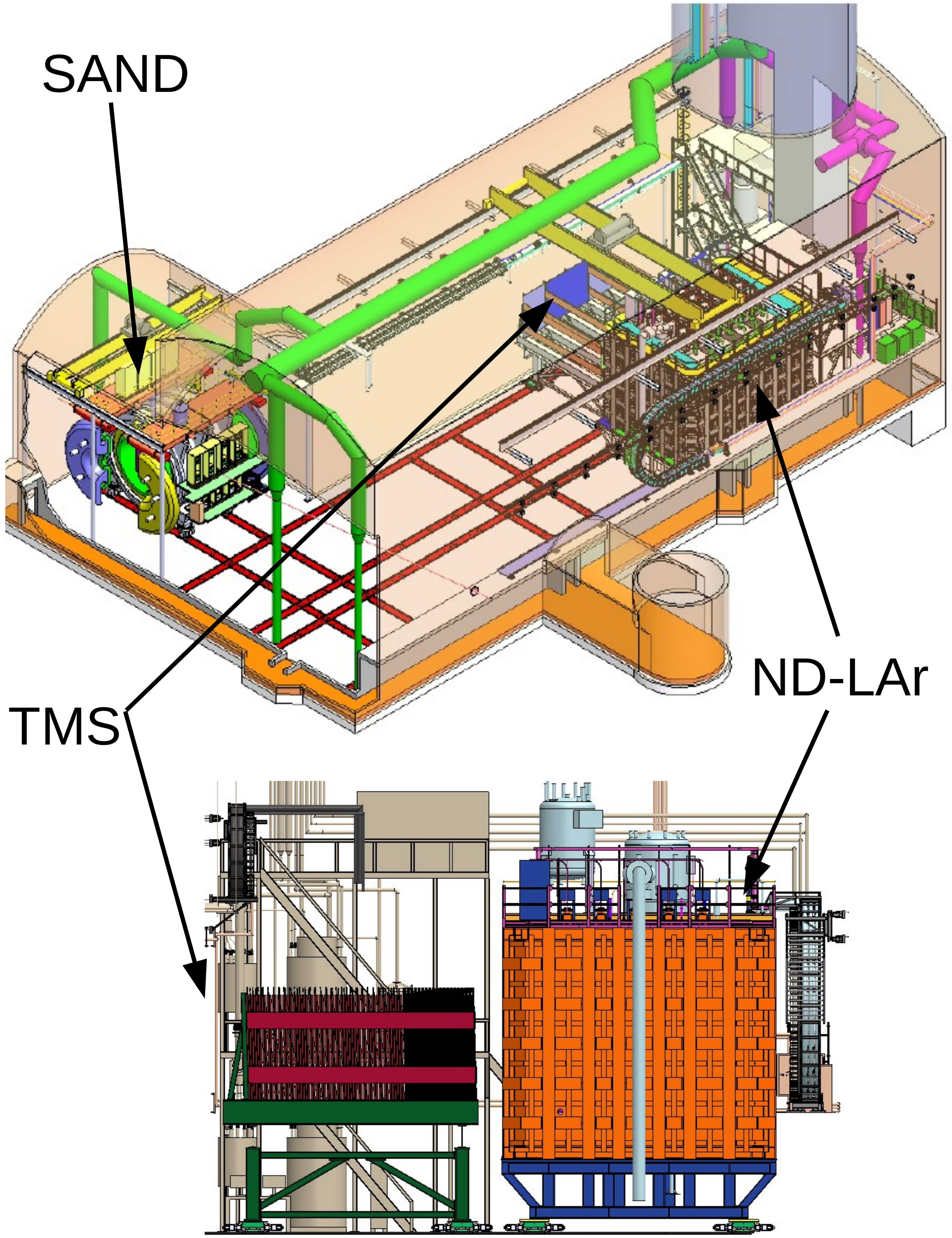
$$\sin^2\theta_{23} \approx 0.5$$

SNO determined $\nu_1 : \nu_2$ Mass Ordering

Fermilab New Accelerator :

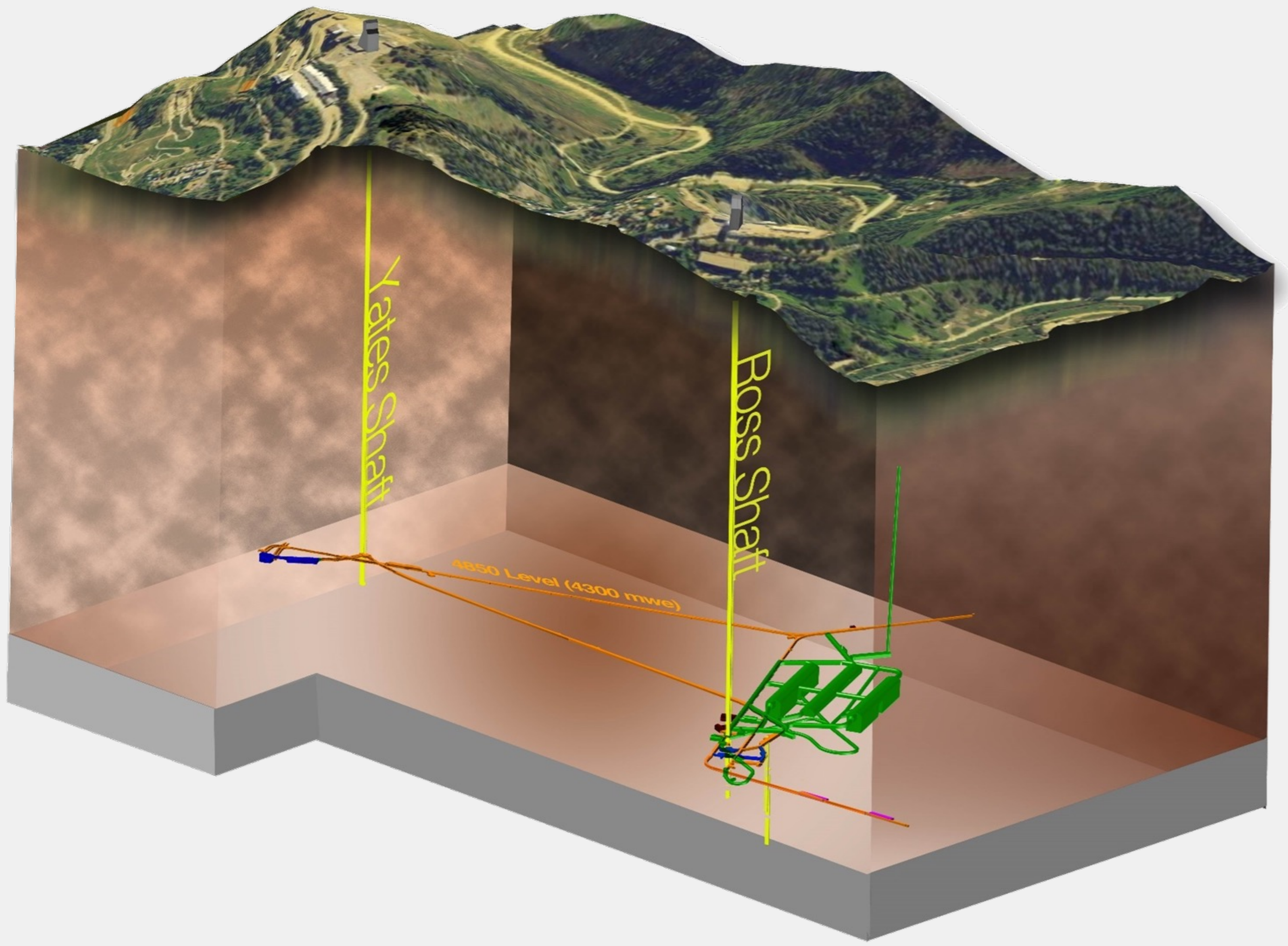
New Beamline :

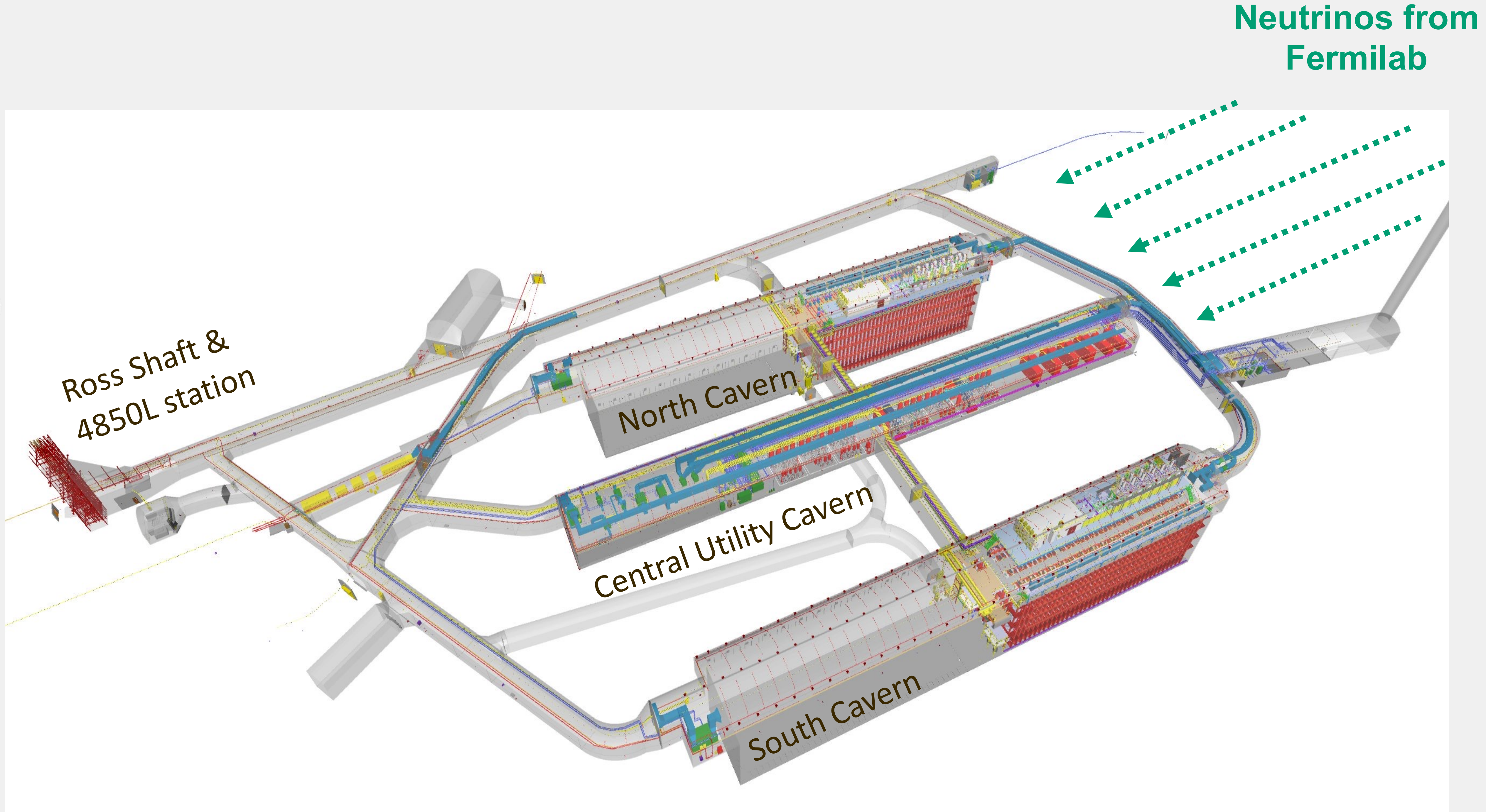




Sophisticated Near Detector:

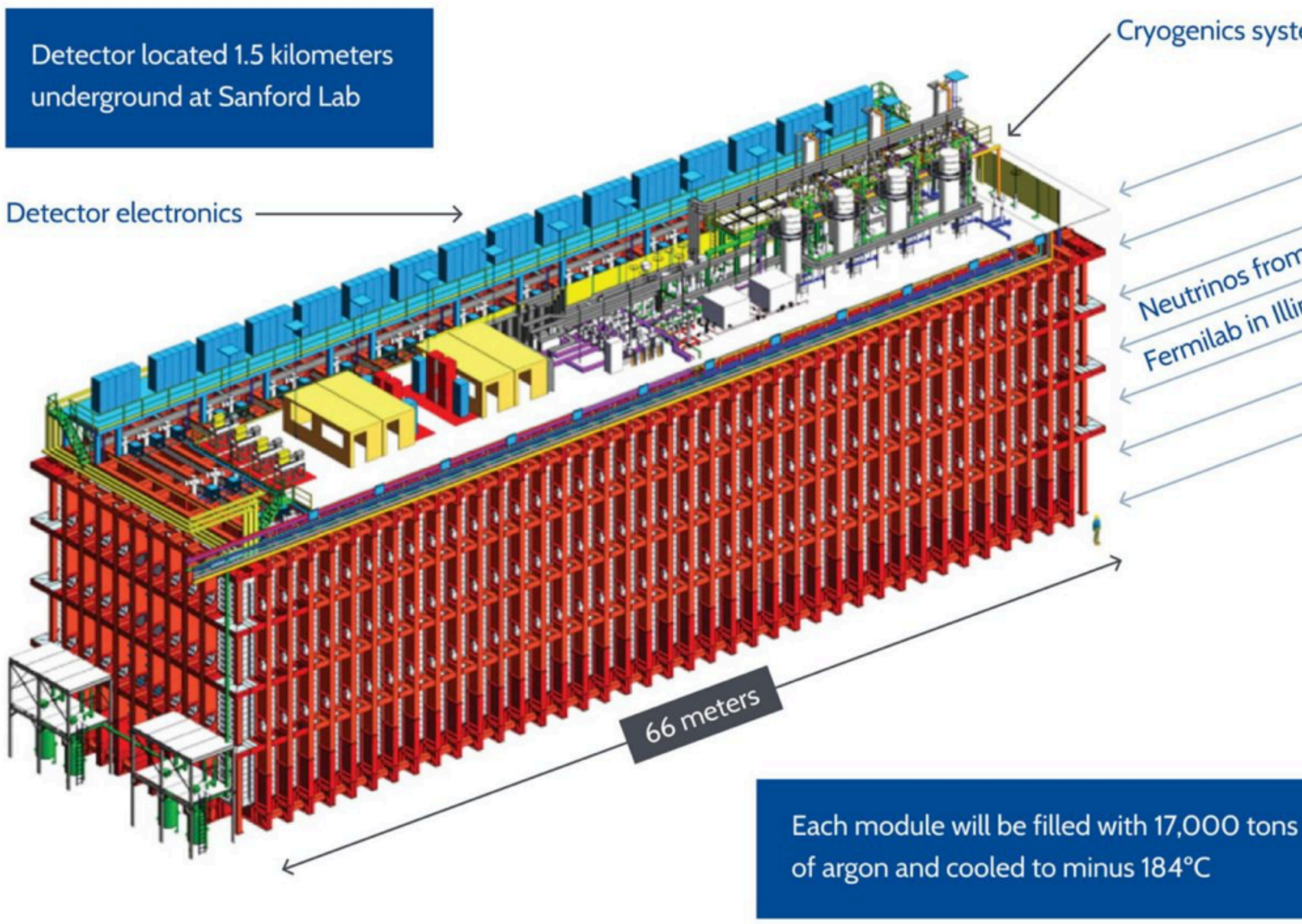
LBNF Far Site at SURF (South Dakota)



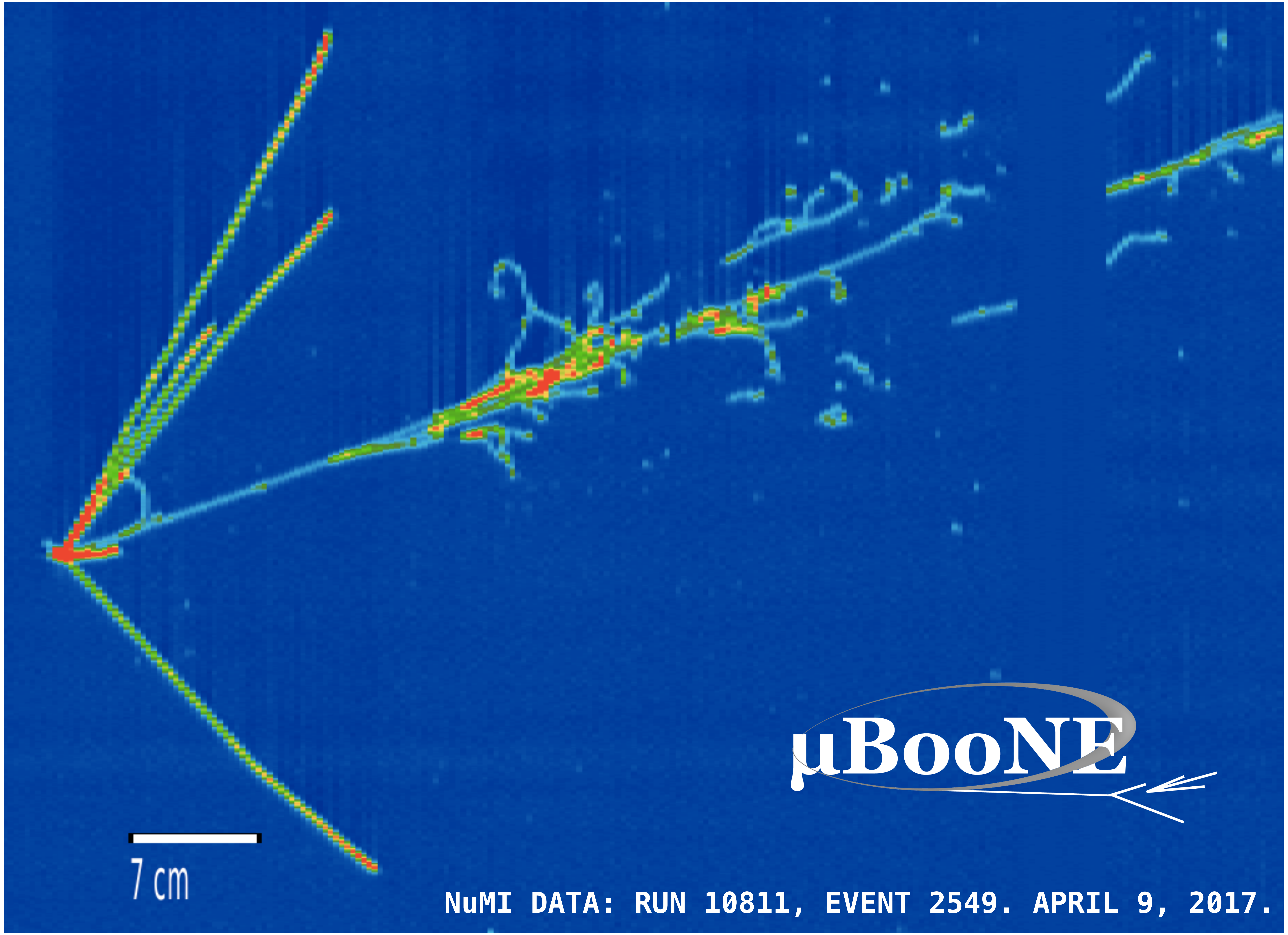


Deepest of the three detectors, 1500 m


JLG ULTRA BOOM


Detector located 1.5 kilometers underground at Sanford Lab
Detector electronics
Cryogenics syst
Neutrinos from
Fermilab in Illi
66 meters
Each module will be filled with 17,000 tons of argon and cooled to minus 184°C

Start 2031



7 cm
μBooNE
NuMI DATA: RUN 10811, EVENT 2549. APRIL 9, 2017.

Neutrino Oscillation Exp.

Only BSM physics Known to be Accessible !

- Mass Pattern (Ordering)
- Flavor Pattern (Octant)
- CP Violation
- New Physics

} Inter-connected

For Dirac vs Majorana nature other types of experiment are required.

Summary:

- Eigenvalues given by solutions to Det$[\lambda I - H] = 0$

- Eigenvectors obtained from: $$v_{\alpha i} v^*_{\beta i} = \frac{\text{Adjg}[\lambda I - H]_{\alpha\beta}}{\text{Det}'[\lambda I - H]}\bigg|_{\lambda=\lambda_i}.$$

only one row or column needed ! Also Local.

(Le Verrier-Faddeev Algorithm)

- Eigenvector-Eigenvalue ID (Jacobi 1834): $$|v_{\alpha i}|^2 = \frac{\Pi_k(\,\lambda_i(H) - \lambda_k(H_\alpha)\,)}{\Pi_k(\,\lambda_i(H) - \lambda_k(H)\,)}$$

now in Wikipedia.

- useful for physicists exploring the physics embedded in small matrices !

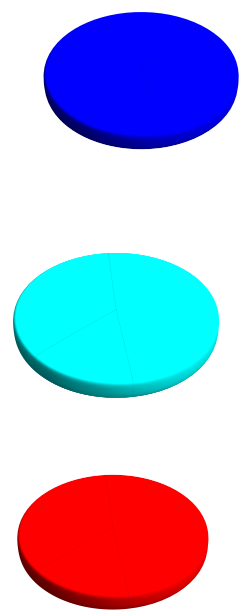
In neutrino physics and HEP, n=3 is a special number.

Extras

Complex Phase δ_{CP} has Significant impact on flavor content of ν_1 and ν_2 mass states :

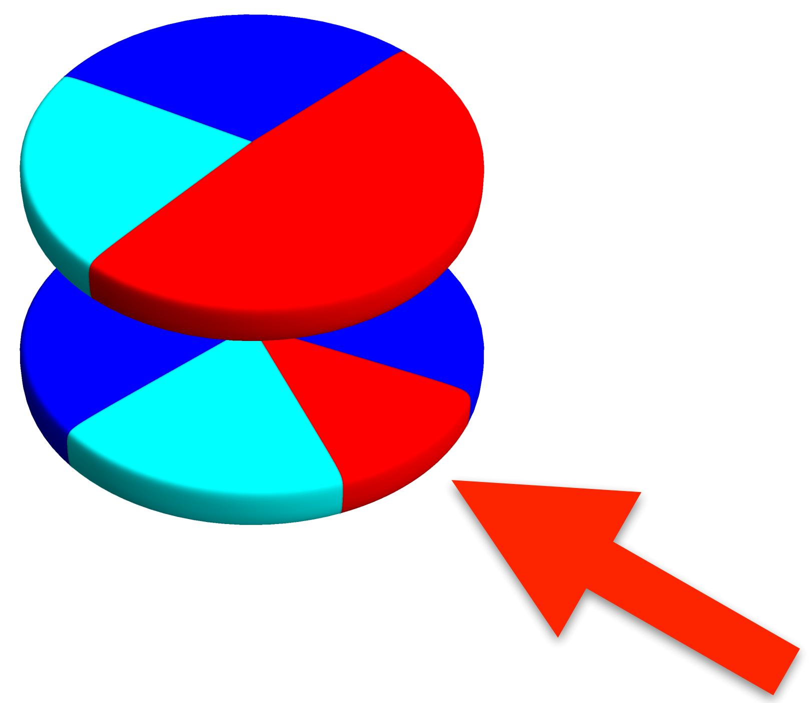
CPC effect

$\nu_e =$ $\nu_\mu =$ $\nu_\tau =$

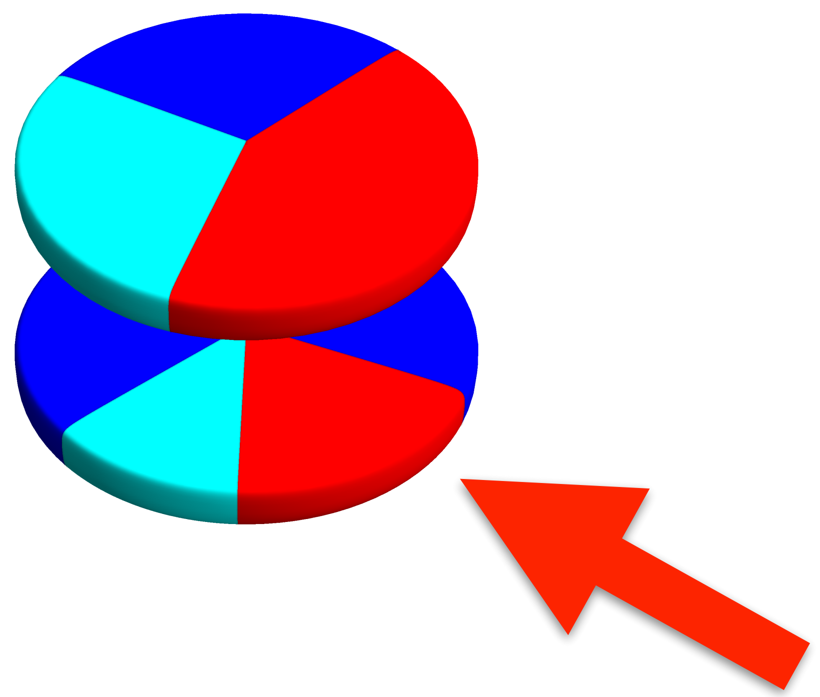


$\sin^2\theta_{23} = 0.6$

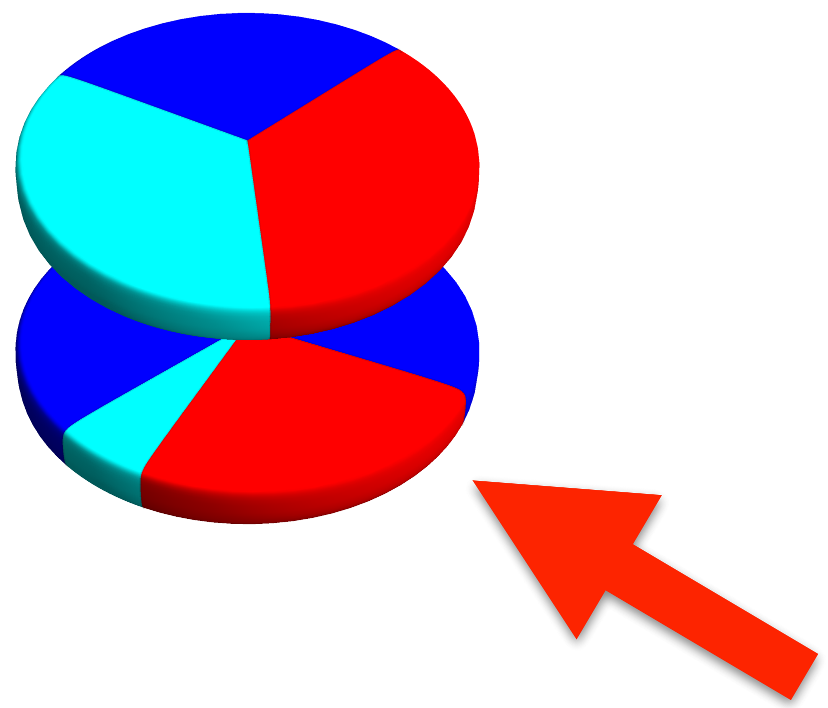
$\delta_{CP} = 0$



$\delta_{CP} = \pm\,\pi/2$



$\delta_{CP} = \pm\,\pi$



$$|U_{\tau 1}|^2 = s_{23}^2 s_{12}^2 + s_{13}^2 c_{12}^2 c_{23}^2 - 2 s_{23} s_{12} s_{13} c_{12} c_{23} \cos\delta_{CP}$$